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TWO-PARAMETER DIFFUSION PROCESSES AND MARTINGALES

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We introduce a class of two-parameter processes which are diffusions on each coordinate and satisfy a particular Markov property related to the partial ordering in R_+^2. These processes can be expressed as solutions of some stochastic integral equations driven by a two-parameter Wiener process and two families of ordinary Brownian motions. This result is based on a characterization of two-parameter martingales with orthogonal increments.

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0. Introduction

Two-parameter martingales with orthogonal increments have been studied by Zakai in [10]. It was shown that martingales with orthogonal increments, satisfying some regularity conditions, can be represented as stochastic integrals of a two-parameter Wiener process. The first part of this paper is devoted to proof some sufficient conditions for a square integrable continuous martingale to have orthogonal increments. These conditions are based on [3, Lemma 4, p. 244].

In Section 2 we consider a class of two-parameter diffusion processes. These processes satisfy the two-parameter Markov property introduced by Cairoli [1] (which allows to characterize the law of the process by a family of transition probabilities and the distribution on the axes), and we suppose that they are ordinary diffusion processes on each coordinate. First we prove that the diffusion operators on each coordinate D_1 and D_2 commute, and the composition $D_1 \circ D_2$ provides a set of two-parameter diffusion coefficients in the sense of [6].

In the last part of this section we obtain the main result of the paper: Under some additional diffusion conditions, a two-parameter diffusion process can be represented as a solution of a stochastic integral equation determined by two families of ordinary Brownian motions indexed by R_+ and a two-parameter Wiener process. As a consequence, this class of process includes the diffusion processes studied by Korezlioglu and Mazziotto in [5].

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1. Two-parameter martingales with orthogonal increments

Denote by R^2_+ the positive quadrant of the plane with the usual order $(s_1, t_1) \leq (s_2, t_2)$ if and only if $s_1 \leq s_2$ and $t_1 \leq t_2$. We will write $(s_1, t_1) < (s_2, t_2)$ if $s_1 < s_2$ and $t_1 < t_2$. If $z_1 < z_2$, $(z_1, z_2]$ will represent the rectangle $\{z \in R^2_+ : z_1 < z \leq z_2\}$. Set $R_z = [(0,0), z]$ for all $z \in R^2_+$, and denote by E_0 the set $\{(s,t) \in R^2_+ : s = 0 \text{ or } t = 0\}$.

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space and consider an increasing family $\{\mathscr{F}_z, z \in R^2_+\}$ of sub-σ-fields of $\mathscr{F}$, satisfying the usual properties:

Property (a). $\mathscr{F}_{00}$ includes the nul sets of $\mathscr{F}$,

Property (b). $\mathscr{F}_z$ is right continuous, and

Property (c). [2, condition F4] for every $z = (s,t) \in R^2_+$, $\mathscr{F}^1_z$ and $\mathscr{F}^2_z$ are conditionally independent given $\mathscr{F}_z$, where $\mathscr{F}^1_z = \bigvee_{y \geq 0} \mathscr{F}_{sy}$ and $\mathscr{F}^2_z = \bigvee_{x \geq 0} \mathscr{F}_{xt}$.

Let $\{M(z), z \in R^2_+\}$ be an $\mathscr{F}_z$-adapted, integrable process null on E_0, and for each $z_1 < z_2$, $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$ we put $M(z_1, z_2] = M(z_2) - M(s_1, t_2) - M(s_2, t_1) + M(z_1)$.

We recall the following definitions:

(1) M_z is a *martingale* if $\mathbf{E}(M(z_2) | \mathscr{F}_{z_1}) = M(z_1)$ for all $z_1 \leq z_2$.

(2) M_z is a *strong martingale* if $\mathbf{E}(M(z_1, z_2] | \mathscr{F}^1_{z_1} \vee \mathscr{F}^2_{z_1}) = 0$ for all $z_1 \leq z_2$.

(3) M_z is a *weak martingale* if $\mathbf{E}(M(z_1, z_2] | \mathscr{F}_{z_1}) = 0$ for all $z_1 \leq z_2$.

Every martingale M_{st} gives rise to the collection of one-parameter martingales $M_s = \{M_{st}, \mathscr{F}^2_t, t \geq 0\}$, $s \geq 0$ being the parameter of the collection. Similarly one can consider the family of one-parameter martingales $M_t = \{M_{st}, \mathscr{F}^1_s, s \geq 0\}$.

For $p \geq 1$, denote by $\mathscr{M}^p$ the space of all martingales satisfying $\sup_z \mathbf{E}(|M_z|^p) < \infty$ (we identify as usual two versions of the same process). Let $\mathscr{M}^p_c$ be the class of continuous martingales in $\mathscr{M}^p$. $\mathscr{M}^2$ is a Hilbert space isometric to $L^2(\Omega, \mathscr{F}, P)$, and $\mathscr{M}^2_c$ is a closed subspace (cf. [2]).

If $M \in \mathscr{M}^2$, $\langle M \rangle$ will denote a right-continuous, $\mathscr{F}_z$-adapted increasing process (in the sense of a positive measure, that means, $\langle M \rangle (z_1, z_2] \geq 0$ for all $z_1 < z_2$), nul on E_0 and such that $M^2 - \langle M \rangle$ is a weak martingale. Following [10], $\mathscr{M}^2_{cc}$ will represent the class of all martingales in $\mathscr{M}^2_c$ for which (a) either $\sup_z \mathbf{E}(M^4_z) < \infty$ or else M is locally L^4 bounded, and (b) $\langle M \rangle$ is sample continuous.

We will say that a martingale M_z of $\mathscr{M}^2$ has 1-*orthogonal increments* if one of the following equivalent conditions is true:

(a) $M_{t_2} - M_{t_1}$ and M_{t_1} are orthogonal one-parameter martingales for all $t_1 < t_2$.

(b) If $(z_1, z'_1] \cap (z_2, z'_2] = \emptyset$, $z_i = (s_i, t_i)$, $i = 1, 2$, then

$$\mathbf{E}(M(z_1, z'_1] M(z_2, z'_2] | \mathscr{F}^1_{s_1 \wedge s_2}) = 0.$$

Martingales with 2-orthogonal increments are defined in an analogous way, and we will say that M_z has orthogonal increments if it has i-orthogonal increments for $i = 1, 2$.

The following characterization of two-parameter martingales with orthogonal increments is due to Zakai (cf. [10]).

1.1. Proposition. *A martingale $M \in \mathcal{M}^2_{cc}$ has orthogonal increments if and only if for all $(s_1, t_1) < (s_2, t_2)$ we have*

$$\langle M_{s_2} - M_{s_1}\rangle_{t_2} - \langle M_{s_2} - M_{s_1}\rangle_{t_1} = \langle M_{t_2} - M_{t_1}\rangle_{s_2} - \langle M_{t_2} - M_{t_1}\rangle_{s_1}.$$

A two-parameter Wiener process $W = \{W_z, z \in R^2_+\}$ is a Gaussian zero mean separable process with covariance function $\mathbf{E}(W(s_1, t_1)W(s_2, t_2)) = (s_1 \wedge s_2)(t_1 \wedge t_2)$. The increasing family of σ-fields generated by W and the nul sets of $\mathcal{F}$ satisfies the usual Properties (a), (b) and (c), and W_z is a strong martingale with respect to this family.

It is immediate that every strong martingale of $\mathcal{M}^2$ has orthogonal increments. Reciprocally, we have the following result (see [10]).

1.2. Proposition. *If $M \in \mathcal{M}^2_{cc}$ is a martingale with orthogonal increments, and $\langle M\rangle$ has a predictable derivative $\alpha(z)$ with respect to the Lebesgue measure in the plane, then there exists a Brownian motion W_z (modifying, if it is necessary, the original probability space) such that $M_z = \int_{R_z} \sqrt{\alpha(\zeta)}\, \mathrm{d}W_\zeta$.*

Moreover, if $\mathcal{F}_z$ is the σ-field generated by a two-parameter Wiener process, then every martingale of $\mathcal{M}^2$ with orthogonal increments is strong, and $\mathcal{M}^2 = \mathcal{M}^2_{cc}$.

The aim of this section is to set up some sufficient conditions for a martingale of $\mathcal{M}^2$ to have orthogonal increments.

For any $z = (s, t) \in R^2_+$, $h > 0$ and $k > 0$ set

$$\Delta^1_h(z) = ((s, 0), (s+h, t)], \qquad \Delta^2_k(z) = ((0, t), (s, t+k)],$$

$$\Delta_{hk}(z) = ((s, t), (s+h, t+k)].$$

The dependence on z will be omitted if there is no danger of confusion. We recall the following result [3, pp. 244–245].

1.3. Lemma. *Let $\{M_t, t \geq 0\}$ be an integrable L^1-continuous process adapted to an increasing family of σ-fields $\{\mathcal{F}_t, t \geq 0\}$. Suppose*

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbf{E}[|\mathbf{E}[M(t+h) - M(t) \mid \mathcal{F}_t]|] = 0. \tag{1.1}$$

Then $\{M_t, \mathcal{F}_t, t \geq 0\}$ is a martingale.

As an immediate consequence of this lemma, we have the following.

1.4. Lemma. *Let M_z be a martingale of $\mathcal{M}^2$ such that for any $t \geq 0$, M_t is L^1-continuous. Suppose*

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbf{E}[|\mathbf{E}[M(\Delta_{hk}(z)) M(\Delta^1_h(z)) \mid \mathcal{F}^1_z]|] = 0. \tag{1.2}$$

Then M_z has 1-orthogonal increments.

Next we are going to prove some sufficient conditions that involve conditional expectations with respect to $\mathscr{F}_z$.

1.5. Lemma. *Let M_z be a martingale of $\mathcal{M}_c^4$, such that*
(a) *for all $z=(s,t)\in \boldsymbol{R}_+^2$ and $\sigma\in[0,s]$*

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbf{E}[M(\Delta_{hk}(z))M(\Delta_h^1(z))M(\Delta_k^2(\sigma,t))\,|\,\mathscr{F}_z]=0 \quad \textit{in probability}, \tag{1.3}$$

and (b) *for all $t_1\leqslant t_2$ the one-parameter increasing process $\langle M_{t_2}-M_{t_1}\rangle(s)$ has a continuous derivative $b_{t_1t_2}(s)$ satisfying* $\sup_s \mathbf{E}(|b_{t_1t_2}(s)|^2)<\infty$. *Then M_z has 1-orthogonal increments.*

Proof. Fix $z=(s,t)\in\boldsymbol{R}_+^2$ and $k>0$. We are going to show that condition (1.2) holds. Consider the continuous bounded variation process

$$B_{t,t+k}(s)=\langle M_t, M_{t+k}-M_t\rangle(s).$$

For all $\omega\in\Omega$ we have

$$\lim_{h\downarrow 0}\frac{1}{h}[B_{t,t+k}(s+h)-B_{t,t+k}(s)]=N_{st}(k) \tag{1.4}$$

where

$$N_{st}(k)=\tfrac{1}{2}[b_{0,t+k}(s)-b_{0,t}(s)-b_{t,t+k}(s)].$$

Then

$$\sup_{h>0}\frac{1}{h^2}\mathbf{E}[|B_{t,t+k}(s+h)-B_{t,t+k}(s)|^2]\leqslant \sup_{h>0}\frac{1}{h}\mathbf{E}\left[\int_s^{s+h}N_{\sigma t}(k)^2\,\mathrm{d}\sigma\right]$$

$$\leqslant \sup_s \mathbf{E}(N_{st}(k)^2)<\infty \tag{1.5}$$

and

$$\mathbf{E}[M(\Delta_{hk}(z))M(\Delta_h^1(z))\,|\,\mathscr{F}_z^1]=\mathbf{E}[B_{t,t+k}(s+h)-B_{t,t+k}(s)\,|\,\mathscr{F}_z^1]. \tag{1.6}$$

From (1.4), (1.5) and (1.6), we obtain, in the L^2 sense,

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbf{E}[M(\Delta_{hk}(z))M(\Delta_h^1(z))\,|\,\mathscr{F}_z^1]=N_{st}(k). \tag{1.7}$$

Thus, it suffices to prove that $N_{st}(k)=0$. Define, for each $s\geqslant 0$, the closed subspace of $L^2(\Omega,\mathscr{F},P)$,

$$H_s=L^2\{1_A(M_{\sigma,t+k}-M_{\sigma t}),\ \sigma\in[0,s],\ A\in\mathscr{F}_{st}\}.$$

From (1.3) and (1.7) we deduce that $N_{st}(k)$ is an element of $L^2(\Omega, \mathscr{F}, P)$ orthogonal to H_s. Indeed,

$$\mathbf{E}[1_A(M_{\sigma,t+k} - M_{\sigma t})N_{st}(k)] =$$

$$= \mathbf{E}\left[1_A(M_{\sigma,t+k} - M_{\sigma t}) \lim_{h\downarrow 0} \frac{1}{h} \mathbf{E}[M(\Delta_{hk}(z))M(\Delta_h^1(z)) \mid \mathscr{F}_z^1]\right]$$

$$= \mathbf{E}\left[1_A \lim_{h\downarrow 0} \frac{1}{h} \mathbf{E}[M(\Delta_k^2(\sigma, t))M(\Delta_{hk}(z))M(\Delta_h^1(z)) \mid \mathscr{F}_z]\right] = 0.$$

Furthermore, $N_{st}(k)$ belongs to H_{s+h} for all $h > 0$. In fact, due to the L^2 convergence in (1.4) we only have to check that $B_{t,t+k}(s+h) - B_{t,t+k}(s) \in H_{s+h}$ for all $h > 0$.

Let $\Pi_n = \{0 = s_0^n < s_1^n < \cdots < s_{k_n}^n = s+h\}$ be a decreasing sequence of partitions of $[0, s+h]$ whose norms converge to zero. Then, due to $M \in \mathscr{M}_c^4$, we have in the L^2 sense

$$B_{t,t+k}(s+h) = \langle M_t, M_{t+k} - M_t \rangle (s+h)$$

$$= \lim_n \sum_{i=1}^{k_n} (M_{s_i^n,t+k} - M_{s_i^n,t} - M_{s_{i-1}^n,t+k} + M_{s_{i-1}^n,t})(M_{s_i^n,t} - M_{s_{i-1}^n,t}).$$

Therefore, $B_{t,t+k}(s+h)$ belongs to H_{s+h}, and this is also true for $B_{t,t+k}(s)$. We conclude that $\{N_{st}(k), s \geq 0\}$ is a continuous square integrable process such that $N_{st}(k)$ is orthogonal to $N_{s't}(k)$ if $s \neq s'$. So, $N_{st}(k) = 0$. $\square$

1.6. Proposition. *Let M_z be a martingale of $\mathscr{M}_c^4$ satisfying the following conditions:*

(a) *For all $(s,t) \in R_+^2$, $\sigma \in [0,s]$ and $\tau \in [0,t]$*

$$\lim_{h,k\downarrow 0} \frac{1}{hk} \mathbf{E}[|\mathbf{E}[M(\Delta_{hk}(s,t))M(\Delta_h^1(s,\tau))M(\Delta_k^2(\sigma,t)) \mid \mathscr{F}_{st}]|] = 0. \tag{1.8}$$

(b) *For all $s_1 \leq s_2$ and $t_1 \leq t_2$ the one-parameter increasing processes $\langle M_{s_2} - M_{s_1}\rangle(t)$ and $\langle M_{t_2} - M_{t_1}\rangle(s)$ have continuous derivatives bounded in L^2.*

Then M_z has orthogonal increments.

Proof. With the notation of Lemma 1.5, note first that $\{N_{st}(k), \mathscr{F}_{t+k}^2, k \geq 0\}$ is a martingale. Indeed, if $0 < k' < k$,

$$\mathbf{E}[N_{st}(k) \mid \mathscr{F}_{s,t+k'}^2] =$$

$$= \mathbf{E}\left[\lim_{h\downarrow 0} \frac{1}{h} \mathbf{E}[M(\Delta_{hk}(z))M(\Delta_h^1(z)) \mid \mathscr{F}_z^1] \mid \mathscr{F}_{s,t+k'}^2\right]$$

$$= \lim_{h\downarrow 0} \frac{1}{h} \mathbf{E}[M(\Delta_{hk}(z))M(\Delta_h^1(z)) \mid \mathscr{F}_{s,t+k'}]$$

$$= \lim_{h\downarrow 0} \frac{1}{h} \mathbf{E}[M(\Delta_{hk'}(z))M(\Delta_h^1(z)) \mid \mathscr{F}_{s,t+k'}^1] = N_{st}(k').$$

Using Lemma 1.3 we are going to prove that $N_{st}(k)$ and $M_{\sigma,t+k}-M_{\sigma t}$, for all $\sigma\in[0,s]$, are $\mathscr{F}^2_{s,t+k}$-adapted orthogonal martingales; that means, their product is a martingale. In fact, if $t<t+k_1<t+k_1+k$, $t+k_1=\tau$, we have

$$\lim_{k\downarrow 0}\frac{1}{k}\mathbf{E}[|\mathbf{E}[(N_{st}(\tau+k)-N_{st}(\tau))(M_{\sigma,\tau+k}-M_{\sigma\tau})\,|\,\mathscr{F}^2_{s\tau}]|]=$$

$$=\lim_{k\downarrow 0}\frac{1}{k}\mathbf{E}\left[\left|\lim_{h\downarrow 0}\frac{1}{h}\mathbf{E}[M(\Delta_{hk}(s,\tau))M(\Delta^1_h(s,t))M(\Delta^2_k(\sigma,\tau))\,|\,\mathscr{F}_{s\tau}]\right|\right]=0.$$

Therefore, by Lemma 1.3

$$\mathbf{E}[N_{st}(k)(M_{\sigma,t+k}-M_{\sigma t})\,|\,\mathscr{F}_{st}]=0,$$

and Lemma 1.5 implies that M_z has 1-orthogonal increments. By a similar argument we conclude that M_z has orthogonal increments. □

1.7. Remarks. (1) As a consequence of Lemma 1.5, a martingale M_z of $\mathscr{M}^4_c$ verifying condition (b) (that means, $\langle M_{t_2}-M_{t_1}\rangle(s)$ has a continuous derivative bounded in L^2, for all $t_1<t_2$) has 1-orthogonal increments if it has 2-orthogonal increments. In fact, condition (a) of Lemma 1.5 holds if M_z has 2-orthogonal increments. This remark and Lemma 1.5 have obvious reciprocal versions.

(2) If $\mathscr{F}_z$ is the increasing family of σ-fields generated by a two-parameter Wiener process W_z, then every martingale of $\mathscr{M}^2$ with 2-(or 1-)orthogonal increments is strong, and this class of martingales is properly included in the family of martingales with path independent variation (cf. [7]).

2. Two-parameter diffusion processes

In this section $\{X(z),\, z\in R^2_+\}$ will be a sample continuous process vanishing on E_0, and we will denote by $\mathscr{F}_z$ the increasing family of σ-fields generated by X_z and the nul sets of $\mathscr{F}$.

2.1. Definition. $X(z)$ will be called a *Markov process* if for all $(s_1,t_1)<(s_2,t_2)$ and B a Borel subset of R, we have

$$\mathbf{P}\{X(s_2,t_2)\in B\,|\,\mathscr{F}^1_{s_1t_1}\vee\mathscr{F}^2_{s_1t_1}\}=\mathbf{P}\{X(s_2,t_2)\in B\,|\,X(s_1,t_2),X(s_1,t_1),X(s_2,t_1)\}. \tag{2.1}$$

This Markov property was introduced in [1], and it was observed that the distribution of $\{X(z),\, z\in R^2_+\}$ is determined by the following transition probabilities,

$$P_{z_1z_2}(x_1,x,x_2,B)=\mathbf{P}\{X(z_2)\in B\,|\,X(s_1,t_2)=x_1,X(z_1)=x,X(s_2,t_1)=x_2\}, \tag{2.2}$$

where $z_1=(s_1,t_1)<z_2=(s_2,t_2)$ and $(x_1,x,x_2)\in R^3$.

Henceforth we will assume that $\{P_{z_1z_2}(\bar{x}, B), z_1 < z_2\}$ is a family of transition probabilities, with $\bar{x} \in R^3$ and B a Borel subset of R, satisfying the following extended versions of the Chapman–Kolmogorov equations:

$$P_{z_1z_2}(x_1, x, x_2, B) = \int_R P_{z_1(\sigma,t_2)}(x_1, x, \xi, d\xi_1) P_{(\sigma,t_1)z_2}(\xi_1, \xi, x_2, B), \tag{2.3}$$

$$P_{z_1z_2}(x_1, x, x_2, B) = \int_R P_{z_1(s_2,\tau)}(\eta, x, x_2, d\eta_1) P_{(s_1,\tau)z_2}(x_1, \eta, \eta_1, B), \tag{2.4}$$

where $s_1 < \sigma < s_2$ and $t_1 < \tau < t_2$.

If we define

$$P_1^t(s, x, s+h, B) = P_{(s,0)(s+h,t)}(x, 0, 0, B)$$

and

$$P_2^s(t, x, t+k, B) = P_{(0,t)(s,t+k)}(0, 0, x, B)$$

for any $s \geq 0$ and $t \geq 0$, then P_1^t and P_2^s are the transition probabilities of the one-parameter Markov processes $\{X_{st}, s \geq 0\}$ and $\{X_{st}, t \geq 0\}$, respectively.

We will also use the fact that the σ-fields $\mathscr{F}_z$ generated by a Markov process satisfy [2, property F4] (cf. [4]).

We are going to suppose that X_z is a diffusion process on each coordinate, and under some additional regularity conditions we will obtain X_z as a solution of a partial stochastic differential equation.

An equivalent Markov property has been studied by Korezlioglu and Lefort [4], and the existence and uniqueness theorems for a system of stochastic differential equations which gives rise to a similar class of diffusion process have been proved by Korezlioglu and Mazziotto [5].

2.2. Definition. A Markov process $X(z)$ will be called a *diffusion process*, provided there exist continuous functions $a_1(s, t, x)$, $B_1(s, t, x)$, $a_2(s, t, x)$, $B_2(s, t, x)$ defined on $R_+^2 \times R$ such that for any $\varepsilon > 0$ the following conditions are satisfied:

$$\int_{|y-x|>\varepsilon} P_1^t(s, x, s+h, dy) = o(h), \tag{2.5}$$

$$\int_{|y-x|>\varepsilon} (y-x) P_1^t(s, x, s+h, dy) = a_1(s, t, x)h + o(h), \tag{2.6}$$

$$\int_{|y-x|>\varepsilon} (y-x)^2 P_1^t(s, x, s+h, dy) = B_1(s, t, x)h + o(h), \tag{2.7}$$

$$\int_{|y-x|>\varepsilon} P_2^s(t, x, t+k, dy) = o(k), \tag{2.8}$$

$$\int_{|y-x|>\varepsilon} (y-x) P_2^s(t, x, t+k, dy) = a_2(s, t, x)k + o(k), \tag{2.9}$$

$$\int_{|y-x|>\varepsilon} (y-x)^2 P_2^s(t, x, t+k, \mathrm{d}y) = B_2(s, t, x)k + \mathrm{o}(k), \tag{2.10}$$

$$\int_{|y-x_1-x_2+x|>\varepsilon} P_{(s,t)(s+h,t+k)}(x_1, x, x_2, \mathrm{d}y) P_1^t(s, x, s+h, \mathrm{d}x_2) P_2^s(t, x, t+k, \mathrm{d}x_1)$$
$$= \mathrm{o}(hk), \tag{2.11}$$

being $h, k > 0$.

Note that (2.11) means $\mathbf{P}\{|X(\Delta_{hk}(s,t))| > \varepsilon \mid \mathcal{F}_{st}\} = \mathrm{o}(hk)$.

Let $X(z)$ be a diffusion process. Henceforth the parameter set will be $I = [0, S] \times [0, T]$. We are going to introduce the following hypotheses.

Hypothesis I. *a_1, a_2, B_1, B_2 have continuous partial derivatives with respect to s and t, and are fourth continuously differentiable in x.*

Hypothesis II. *Conditions* (2.5), (2.8) *and* (2.11) *are satisfied uniformly with respect to $(s, t) \in I$.*

Hypothesis III. *For each compact K, there exist constants l and c such that*

(a) *for $x \in K$,*

$$\left| \int_{|y-x|\leq\varepsilon} (y-x) P_1^t(s, x, s+h, \mathrm{d}y) \right| + \int_{|y-x|\leq\varepsilon} (y-x)^2 P_1^t(s, x, s+h, \mathrm{d}y) \leq lh,$$

$$\left| \int_{|y-x|\leq\varepsilon} (y-x) P_2^s(t, x, t+k, \mathrm{d}y) \right| + \int_{|y-x|\leq\varepsilon} (y-x)^2 P_2^s(t, x, t+k, \mathrm{d}y) \leq lk.$$

(b) $\sup_{|x|>c} P_1^t(s, x, s+h, K) \leq lh$, $\sup_{|x|>c} P_2^s(t, x, t+k, K) \leq lk$ *for all* $(s, t) \in I$.

Under these conditions we know ([3, Theorem 10, pp. 245–246]) that there exist two families of ordinary Brownian motions $\{W_{st}^1, s \geq 0\}$, $\{W_{st}^2, t \geq 0\}$ adapted to the σ-fields $\mathcal{F}_{st}$ (we modify, if necessary, the original probability space), such that

$$W_{st}^1 = \int_0^s a_1(\sigma, t, X_{\sigma t})\,\mathrm{d}\sigma + \int_0^s B_1^{1/2}(\sigma, t, X_{\sigma t}) W^1(\mathrm{d}\sigma, t) \tag{2.12}$$

and

$$W_{st}^2 = \int_0^t a_2(s, \tau, X_{s\tau})\,\mathrm{d}\tau + \int_0^t B_2^{1/2}(s, \tau, X_{s\tau}) W^2(s, \mathrm{d}\tau) \tag{2.13}$$

for all $(s, t) \in I$.

Furthermore $W_{st}^1 - W_{s't}^1$ is independent of $\mathcal{F}_{s't}^1$ for all $s' < s$, and $W_{st}^2 - W_{st'}^2$ is independent of $\mathcal{F}_{st'}^2$ for all $t' < t$.

2.3. Proposition. *Let $X(z)$ be a diffusion process satisfying Hypotheses* I, II *and* III. *The diffusion operators*

$$D_1=\frac{\partial}{\partial s}+a_1\frac{\partial}{\partial x}+\tfrac{1}{2}B_1\frac{\partial^2}{\partial x^2} \quad \textit{and} \quad D_2=\frac{\partial}{\partial t}+a_2\frac{\partial}{\partial x}+\tfrac{1}{2}B_2\frac{\partial^2}{\partial x^2}$$

commute.

Proof. Let $f:R\to R$ be an infinitely differentiable function with compact support. Applying Ito's formula, we have

$$\begin{aligned}&\mathbf{E}[f(X)(\Delta_{hk}(z))\,|\,\mathcal{F}_z]=\\&=\mathbf{E}\left[\int_s^{s+h}[D_1f(\sigma,t+k,X_{\sigma,t+k})-D_1f(\sigma,t,X_{\sigma t})]\,\mathrm{d}\sigma\,|\,\mathcal{F}_z\right]\\&=\mathbf{E}\left[\int_s^{s+h}\int_t^{t+k}D_2D_1f(\sigma,\tau,X_{\sigma\tau})\,\mathrm{d}\sigma\,\mathrm{d}\tau\,|\,\mathcal{F}_z\right].\end{aligned}$$

D_2D_1f is a bounded continuous function. So, by the dominated convergence theorem, we obtain a.s., and in the L^1 sense

$$\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[f(X)(\Delta_{hk}(z))\,|\,\mathcal{F}_z]=D_2D_1f(z,X_z). \tag{2.14}$$

In the same manner this limit can be proved to be equal to $D_1D_2f(z,X_z)$. Due to the continuity of $X(z)$ we have, a.s., $D_2D_1f(z,X_z)=D_1D_2f(z,X_z)$ for all z in I, and therefore, $D_2D_1f(z,x)=D_1D_2f(z,x)$ for all x in the range of X_z. □

We will need an additional hypothesis in order to prove the next proposition.

Hypothesis IV. *For each compact K, there exists a constant l such that if $x,\xi\in K$, then*

$$\left|\int_{\substack{|x_2-x|\le\varepsilon\\|\xi_1-\xi|\le\varepsilon}}(x_2-x)(\xi_1-\xi)P_{(s,\tau)(s+h,t)}(x,\xi,\xi_1,\mathrm{d}x_2)P_1^{\tau}(s,\xi,s+h,\mathrm{d}\xi_1)\right|\le lh$$

and

$$\left|\int_{\substack{|x_1-x|\le\varepsilon\\|\eta_1-\eta|\le\varepsilon}}(x_1-x)(\eta_1-\eta)P_{(\sigma,t)(s,t+k)}(\eta_1,\eta,x,\mathrm{d}x_1)P_2^{\sigma}(t,\eta,t+k,\mathrm{d}\eta_1)\right|\le lk$$

for all $(s,t)\in I$, $\tau\in[0,t)$ and $\sigma\in[0,s)$.

For any random variable Y and for $\varepsilon>0$ we put $Y_\varepsilon=Y1_{\{|Y|\le\varepsilon\}}$.

2.4. Proposition. *Let $X(z)$ be a diffusion process satisfying Hypotheses* I *to* IV. *Then, the functions*

$$a=D_1a_2=D_2a_1,\qquad B=D_2B_1-2B_1\frac{\partial a_2}{\partial x}=D_1B_2-2B_2\frac{\partial a_1}{\partial x},$$

$$c_1=B_1\frac{\partial a_2}{\partial x},\qquad c_2=B_2\frac{\partial a_1}{\partial x},\qquad d=\tfrac{1}{2}B_1\frac{\partial B_2}{\partial x}=\tfrac{1}{2}B_2\frac{\partial B_1}{\partial x}$$

are two-parameter diffusion coefficients of $X(z)$ *(cf.* [6]) *in the following sense,*

$$\mathbf{E}[X(\Delta_{hk}(z))^i X(\Delta_h^1(z))^j X(\Delta_k^2(z))^l \mid \mathcal{F}_z] =$$

$$= \begin{cases} a(z, X_z)hk + o(hk) & \text{if } i=1, j, l=0, & (2.15) \\ B(z, X_z)hk + o(hk) & \text{if } i=2, j, l=0, & (2.16) \\ c_1(z, X_z)hk + o(hk) & \text{if } i=j=1, l=0, & (2.17) \\ c_2(z, X_z)hk + o(hk) & \text{if } i=l=1, j=0, & (2.18) \\ d(z, X_z)hk + o(hk) & \text{if } i=j=l=1. & (2.19) \end{cases}$$

Proof. Fix $z=(s,t)\in I$, $X(z)=x_0$, $C>0$ and let f_1 be an infinitely differentiable function with compact support such that $f_1(x)=x$ if $|x-x_0|\leqslant C$. Define

$$A_{hk\varepsilon} = \{\omega : |X(\Delta_{hk}(z))(\omega)| \leqslant \varepsilon, |X(\Delta_h^1(z))(\omega)| \leqslant \varepsilon, |X(\Delta_k^2(z))(\omega)| \leqslant \varepsilon\}.$$

Suppose $C \geqslant 3\varepsilon$. Then, for all $\omega \in A_{hk\varepsilon}$ such that $X_z(\omega)=x_0$ we have

$$f_1(X)(\Delta_{hk}(z)) = X(\Delta_{hk}(z)), \quad f_1(X)(\Delta_h^1(z)) = X(\Delta_h^1(z))$$
$$\text{and} \quad f_1(X)(\Delta_k^2(z)) = X(\Delta_k^2(z)).$$

Then, from (2.14) we obtain

$$\lim_{h,k\downarrow 0} \frac{1}{hk} \mathbf{E}[f_1(X)(\Delta_{hk}(z)) \mid \mathcal{F}_z] = D_2 D_1 f_1(z, X_z) = a(z, x_0).$$

Therefore, in order to prove (2.15) it suffices to check the following equality,

$$\mathbf{E}[(f_1(X)(\Delta_{hk}(z)) - X(\Delta_{hk}(z))_\varepsilon) 1_{A^c_{hk\varepsilon}} \mid \mathcal{F}_z] = o(hk). \tag{2.20}$$

From (2.5), (2.8), (2.11) and property F4 of the family $\mathcal{F}_z$ [2] we deduce

$$\begin{aligned} \mathbf{P}\{A^c_{hk\varepsilon} \mid \mathcal{F}_z\} = \mathbf{P}\{\{|X(\Delta_{hk})| \leqslant \varepsilon\} \\ \cap [\{|X(\Delta_h^1)| \leqslant \varepsilon, |X(\Delta_k^2)| > \varepsilon\} \\ \cup \{|X(\Delta_h^1)| > \varepsilon, |X(\Delta_k^2)| \leqslant \varepsilon\}] \mid \mathcal{F}_z\} + o(hk). \end{aligned} \tag{2.21}$$

As a consequence, to show (2.20) we have to bound the expression

$$\begin{aligned} &|\mathbf{E}[(f_1(X)(\Delta_{hk}) - X(\Delta_{hk})) 1_{\{|X(\Delta_{hk})| \leqslant \varepsilon, |X(\Delta_h^1)| \leqslant \varepsilon, |X(\Delta_k^2)| > \varepsilon\}} \mid \mathcal{F}_z]| = \\ &= |\mathbf{E}[(f_1(X_{s+h,t+k}) - f_1(X_{s,t+k}) - X_{s+h,t+k} + X_{s,t+k}) \\ &\quad \times 1_{\{|X(\Delta_{hk})| \leqslant \varepsilon, |X(\Delta_h^1)| \leqslant \varepsilon, |X(\Delta_k^2)| > \varepsilon\}} \mid \mathcal{F}_z] \\ &\leqslant \left|\mathbf{E}\left[\left(\int_s^{s+h} D_1 f_1(\sigma, t+k, X_{\sigma,t+k})\, d\sigma\right) 1_{\{|X(\Delta_k^2)| > \varepsilon\}} \mid \mathcal{F}_z\right]\right| \\ &\quad + |\mathbf{E}[\mathbf{E}[(X_{s+h,t+k} - X_{s,t+k}) 1_{\{|X(\Delta_h^1(s,t+k))| \leqslant 2\varepsilon\}} \mid \mathcal{F}_{s,t+k}] 1_{\{|X(\Delta_k^2)| > \varepsilon\}} \mid \mathcal{F}_z]| \\ &\quad + o(hk) \\ &\leqslant \text{const. } h \mathbf{P}\{|X(\Delta_k^2)| > \varepsilon \mid \mathcal{F}_z\} + o(hk) = o(hk). \end{aligned}$$

In the last inequality we have used the fact that $D_1 f_1$ is bounded, and Hypothesis III. The other term given by (2.21) can be treated similarly, and (2.20) holds.

In order to prove (2.17) let f_2 be an infinitely differentiable function with compact support. Then,

$$
\begin{aligned}
&\mathbf{E}[f_1(X)(\Delta_{hk})f_2(X)(\Delta_h^1)\,|\,\mathcal{F}_z]= \\
&\quad=\mathbf{E}\Big[\Big(\int_t^{t+k}(D_2f_1(s+h,\tau,X_{s+h,\tau})-D_2f_1(s,\tau,X_{s\tau}))\,\mathrm{d}\tau\Big)f_2(X)(\Delta_h^1)\,|\,\mathcal{F}_z\Big] \\
&\quad=k\,\mathbf{E}[(D_2f_1(s+h,t,X_{s+h,t})-D_2f_1(s,t,X_{st}))f_2(X)(\Delta_h^1)\,|\,\mathcal{F}_z] \\
&\qquad+\mathbf{E}\Big[\Big(\int_t^{t+k}\Big(\int_t^{\tau}(D_2D_2f_1(s+h,\tau',X_{s+h,\tau'}) \\
&\qquad\qquad-D_2D_2f_1(s,\tau',X_{s\tau'}))\,\mathrm{d}\tau'\Big)\,\mathrm{d}\tau\Big)f_2(X)(\Delta_h^1)\,|\,\mathcal{F}_z\Big] \\
&\quad=k\,\mathbf{E}\Big[\int_s^{s+h}\frac{\partial}{\partial x}(D_2f_1)(\sigma,t,X_{\sigma t})f_2'(X_{\sigma t})B_1(\sigma,t,X_{\sigma t})\,\mathrm{d}\sigma\,|\,\mathcal{F}_z\Big]+\mathrm{o}(hk).
\end{aligned}
$$

Here we have used Hypothesis IV to assert that

$$
\begin{aligned}
&|\mathbf{E}[(D_2D_2f_1(s+h,\tau',X_{s+h,\tau'})-D_2D_2f_1(s,\tau',X_{s\tau'}))f_2(X)(\Delta_h^1)\,|\,\mathcal{F}_z]|\leqslant \\
&\quad\leqslant \text{const.}\,h.
\end{aligned}
$$

Therefore, we obtain

$$
\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[f_1(X)(\Delta_{hk})f_2(X)(\Delta_h^1)\,|\,\mathcal{F}_z]=\Big(\frac{\partial}{\partial x}(D_1f_1)f_2'B_1\Big)(z,X_z), \tag{2.22}
$$

where the convergence is a.s., and L^1, due to the dominated convergence theorem. In particular, we have

$$
\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[f_1(X)(\Delta_{hk})f_1(X)(\Delta_h^1)\,|\,\mathcal{F}_z]=B_1\frac{\partial a_2}{\partial x}(z,X_z)=c_1(z,x_0).
$$

In consequence, to show (2.17) we only have to check the equality

$$
\mathbf{E}[(f_1(X)(\Delta_{hk})f_1(X)(\Delta_h^1)-X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon)1_{A_{hk\varepsilon}^c}\,|\,\mathcal{F}_z]=\mathrm{o}(hk). \tag{2.23}
$$

Using (2.21) it suffices to consider the following two terms:

$$
\begin{aligned}
\text{(i)}\quad &|\mathbf{E}[(f_1(X)(\Delta_{hk})f_1(X)(\Delta_h^1)-X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon)\times \\
&\quad\times 1_{\{|X(\Delta_{hk})|\leqslant\varepsilon,|X(\Delta_h^1)|>\varepsilon,|X(\Delta_k^2)|\leqslant\varepsilon\}}\,|\,\mathcal{F}_z]| \\
&\leqslant|\mathbf{E}[(f_1(X_{s+h,t+k})-f_1(X_{s+h,t})-X_{s,t+k}+X_{st})f_1(X)(\Delta_h^1) \\
&\quad\times 1_{\{|X(\Delta_h^1)|>\varepsilon,|X(\Delta_k^2)|\leqslant\varepsilon\}}\,|\,\mathcal{F}_z]| \\
&\leqslant\Big|\mathbf{E}\Big[\Big(\int_t^{t+k}D_2f_1(s+h,\tau,X_{s+h,\tau})\,\mathrm{d}\tau\Big)f_1(X)(\Delta_h^1)1_{\{|X(\Delta_h^1)|>\varepsilon\}}\,|\,\mathcal{F}_z\Big]\Big| \\
&\quad+|\mathbf{E}[X(\Delta_k^2)1_{\{|X(\Delta_k^2)|\leqslant\varepsilon\}}\,|\,\mathcal{F}_z]\mathbf{E}[f_1(X)(\Delta_h^1)1_{\{|X(\Delta_h^1)|\leqslant\varepsilon\}}\,|\,\mathcal{F}_z]| \\
&\leqslant\text{const.}\,k\,\mathbf{P}\{|X(\Delta_h^1)|>\varepsilon\,|\,\mathcal{F}_z\}=\mathrm{o}(hk).
\end{aligned}
$$

(ii) $$|\mathbf{E}[(f_1(X)(\Delta_{hk})f_1(X)(\Delta_h^1)-X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon)\times$$
$$\times 1_{\{|X(\Delta_{hk})|\le\varepsilon,|X(\Delta_h^1)|\le\varepsilon,|X(\Delta_k^2)|>\varepsilon\}}\,|\,\mathscr{F}_z]|$$
$$\le|\mathbf{E}[(f_1(X_{s+h,t+k})-f_1(X_{s,t+k})-X_{s+h,t+k}+X_{s,t+k})X(\Delta_h^1)$$
$$\times 1_{\{|X(\Delta_{hk})|\le\varepsilon,|X(\Delta_h^1)|\le\varepsilon,|X(\Delta_k^2)|>\varepsilon\}}\,|\,\mathscr{F}_{st}]|$$
$$\le|\mathbf{E}[(f_1'(X_{s,t+k})-1)\mathbf{E}[X(\Delta_h^1(s,t+k))X(\Delta_h^1(s,t))$$
$$\times 1_{\{|X(\Delta_h^1(s,t+k))|\le 2\varepsilon,|X(\Delta_h^1(s,t))|\le\varepsilon\}}\,|\,\mathscr{F}_{s,t+k}]1_{\{|X(\Delta_k^2)|>\varepsilon\}}\,|\,\mathscr{F}_z]|$$
$$+\varepsilon\|f_1''\|_\infty|\mathbf{E}[\mathbf{E}[X(\Delta_h^1(s,t+k))^2 1_{\{|X(\Delta_h^1(s,t+k))|\le 2\varepsilon\}}\,|\,\mathscr{F}_{s,t+k}]$$
$$\times 1_{\{|X(\Delta_k^2)|>\varepsilon\}}\,|\,\mathscr{F}_z]|+\mathrm{o}(hk)$$
$$\le\text{const. } h\,\mathbf{P}\{|X(\Delta_k^2)|>\varepsilon\,|\,\mathscr{F}_z\}+\mathrm{o}(hk)=\mathrm{o}(hk).$$

In the last inequality we have used Hypotheses III and IV. So, (2.17) is proved, and the proof of (2.18) follows the same lines.

Now we are going to prove (2.16). In the same conditions as above, let f_2 be an infinitely differentiable function with compact support such that $f_2(x)=x^2$ if $|x-x_0|\le C$. Then, from (2.14) we have

$$\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[f_2(X)(\Delta_{hk})\,|\,\mathscr{F}_z]=D_1D_2x^2(z,x_0). \tag{2.24}$$

If $C\ge 3\varepsilon$, for all $\omega\in A_{hk\varepsilon}$ such that $X_z(\omega)=x_0$, we obtain

$$f_2(X)(\Delta_{hk})=X^2(\Delta_{hk})=X(\Delta_{hk})^2+2X(\Delta_h^1)X(\Delta_k^2)+2X(\Delta_{hk})X(\Delta_h^1)$$
$$+2X(\Delta_{hk})X(\Delta_k^2)+2X_{st}X(\Delta_{hk}).$$

The following relation can be derived as in the proof of (2.20) and (2.23),

$$\mathbf{E}[(f_2(X)(\Delta_{hk})-X(\Delta_{hk})_\varepsilon^2-2X(\Delta_h^1)_\varepsilon X(\Delta_k^2)_\varepsilon-2X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon$$
$$-2X(\Delta_{hk})_\varepsilon X(\Delta_k^2)_\varepsilon-2X_{st}X(\Delta_{hk})_\varepsilon)1_{A_{hk\varepsilon}^c}\,|\,\mathscr{F}_z]=\mathrm{o}(hk). \tag{2.25}$$

Therefore, from (2.24) and (2.25) we deduce

$$\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[X(\Delta_{hk})_\varepsilon^2\,|\,\mathscr{F}_z]=(D_1D_2x^2-2a_1a_2-2c_1-2c_2-2xa)(z,X_z)$$
$$=B(z,X_z).$$

To prove (2.19), we first compute the following limit, using (2.22),

$$\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[f_2(X)(\Delta_{hk})f_1(X)(\Delta_h^1)\,|\,\mathscr{F}_z]=$$
$$=\Big(\frac{\partial}{\partial x}(D_2f_2)f_1'B_1\Big)(z,X_z)=\Big(\frac{\partial B_2}{\partial x}+2a_2+2x\frac{\partial a_2}{\partial x}\Big)B_1(z,X_z). \tag{2.26}$$

For all $\omega \in A_{hk\varepsilon}$ such that $X_z(\omega)=x_0$, we have

$$f_2(X)(\Delta_{hk})f_1(X)(\Delta_h^1)=X(\Delta_{hk})^2X(\Delta_h^1)+2X(\Delta_h^1)^2X(\Delta_k^2)+2X(\Delta_{hk})X(\Delta_h^1)^2 \\ +2X(\Delta_{hk})X(\Delta_h^1)X(\Delta_k^2)+2X_{st}X(\Delta_{hk})X(\Delta_h^1). \tag{2.27}$$

Then, we can deduce, as above

$$\begin{aligned}\mathbf{E}[(f_2(X)(\Delta_{hk})f_1(X)(\Delta_h^1)-X(\Delta_{hk})_\varepsilon^2X(\Delta_h^1)_\varepsilon-2X(\Delta_h^1)_\varepsilon^2X(\Delta_k^2)_\varepsilon \\ -2X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon^2-2X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon X(\Delta_k^2)_\varepsilon \\ -2X_{st}X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon)1_{A^c_{hk\varepsilon}}|\mathcal{F}_z]=\mathrm{o}(hk).\end{aligned} \tag{2.28}$$

Furthermore,

$$\mathbf{E}[X(\Delta_{hk})_\varepsilon^2X(\Delta_h^1)_\varepsilon\,|\mathcal{F}_z]=\mathrm{o}(hk). \tag{2.29}$$

Indeed, for any $0<\delta<\varepsilon$ we have

$$\begin{aligned}&|\mathbf{E}[X(\Delta_{hk})_\varepsilon^2X(\Delta_h^1)_\varepsilon\,|\mathcal{F}_z]|\leqslant \\ &\quad\leqslant \delta hkB(z,X_z)+\mathrm{o}(hk)+\mathbf{E}[X(\Delta_{hk})_\varepsilon^2X(\Delta_h^1)_\varepsilon\,1_{\{\delta<|X(\Delta_h^1)|\leqslant\varepsilon\}}\,|\mathcal{F}_z]| \\ &\quad\leqslant \delta hkB(z,X_z)+\mathrm{o}(hk) \\ &\qquad+\mathbf{E}[2[X(\Delta_k^2)_\varepsilon^2+X(\Delta_k^2(s+h,t))_{2\varepsilon}^2]\varepsilon\,1_{\{\delta<|X(\Delta_h^1)|\leqslant\varepsilon\}}\,|\mathcal{F}_z] \\ &\quad=\delta hkB(z,X_z)+\mathrm{o}(hk).\end{aligned}$$

So, being $\delta>0$ arbitrary we conclude that (2.29) holds.

We can also prove that

$$\mathbf{E}[X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon^2|\mathcal{F}_z]=\mathrm{o}(hk). \tag{2.30}$$

In fact, let f_1 be an infinitely differentiable function with compact support such that $f_1(x)=x$ if $|x-x_0|\leqslant C$, where $0<C<3\varepsilon$. Then, for all $\omega\in A_{hk\varepsilon}$ such that $X_z(\omega)=x_0$, we have

$$f_1(X)(\Delta_{hk})f_1(X)(\Delta_h^1)^2=X(\Delta_{hk})X(\Delta_h^1)^2,$$

and the following relation holds,

$$\mathbf{E}[(f_1(X)(\Delta_{hk})f_1(X)(\Delta_h^1)^2-X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon^2)1_{A^c_{hk\varepsilon}}|\mathcal{F}_z]=\mathrm{o}(hk).$$

So, it suffices to compute

$$\begin{aligned}&|\mathbf{E}[f_1(X)(\Delta_{hk})f_1(X)(\Delta_h^1)^2|\mathcal{F}_z]|= \\ &\quad=\left|\mathbf{E}\left[\left(\int_t^{t+k}(D_2f_1(s+h,\tau,X_{s+h,\tau})-D_2f_1(s,\tau,X_{s\tau}))\,\mathrm{d}\tau\right)f_1(X)(\Delta_h^1)^2|\mathcal{F}_z\right]\right| \\ &\quad=k|\mathbf{E}[(D_2f_1(s+h,t,X_{s+h,t})-D_2f_1(s,t,X_{st}))f_1(X)(\Delta_h^1)^2|\mathcal{F}_z]|+\mathrm{o}(hk) \\ &\quad=\mathrm{o}(hk).\end{aligned}$$

Finally, from (2.26), (2.27), (2.28), (2.29) and (2.30) we conclude

$$\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon X(\Delta_k^2)_\varepsilon\,|\,\mathcal{F}_z]=$$

$$=\tfrac{1}{2}\left(\frac{\partial B_2}{\partial x}+2a_2+2x\frac{\partial a_2}{\partial x}\right)B_1-2B_1a_2-2xc_1(z,X_z)=d(z,X_z). \quad \square$$

The two-parameter processes W_{st}^1 and W_{st}^2 introduced in (2.12) and (2.13) have measurable versions. In fact, if $\hat{W}_{st}$ is a two-parameter Wiener process independent of X_{st}, we can set, for instance,

$$W_{st}^1=\int_0^s B_1^{-1/2}(\sigma,t,X_{\sigma t})\xi^1(\mathrm{d}\sigma,t)+\int_0^s 1_{\{B_1(\sigma,t,X_{\sigma t})=0\}}t^{-1}\hat{W}(\mathrm{d}\sigma,t)$$

where ξ_{st}^1 is the one-parameter continuous local martingale (with respect to s) $X_{st}-\int_0^s a_1(\sigma,t,X_{\sigma t})\,\mathrm{d}\sigma$.

Then, the measurability of W_{st}^1 follows from the results of Stricker and Yor [8].

Let $\{f_{st},(s,t)\in I\}$ be a measurable, $\mathcal{F}_{st}^1$-adapted process, such that

$$\mathbf{P}\left\{\int_0^T\int_0^S f_{st}^2\,\mathrm{d}s\,\mathrm{d}t<\infty\right\}=1.$$

Then, for such processes the following iterated integral can be defined as

$$\int_0^T\left(\int_0^S f_{st}W^1(\mathrm{d}s,t)\right)\mathrm{d}t. \tag{2.31}$$

Indeed, for all $t\in[0,T]$ except in a set of Lebesgue measure zero the stochastic integral $\int_0^s f_{\sigma t}W^1(\mathrm{d}\sigma,t)$ exists. We can choose versions of these integrals, such that they define a measurable function on $[0,S]\times[0,T]\times\Omega$ (cf. [8]).

Thus, in order to define (2.31) it suffices to verify that

$$\mathbf{P}\left\{\int_0^T\left|\int_0^S f_{st}W^1(\mathrm{d}s,t)\right|\mathrm{d}t<\infty\right\}=1. \tag{2.32}$$

For any $N>0$, set $f_{st}^N(\omega)=f_{st}(\omega)1_{\{\int_0^t\int_0^s f_{\sigma\tau}^2(\omega)\,\mathrm{d}\sigma\,\mathrm{d}\tau\leq N\}}$. If $C>0$, we have

$$\mathbf{P}\left\{\int_0^T\left|\int_0^S f_{st}W^1(\mathrm{d}s,t)\right|\mathrm{d}t>C\right\}\leq$$

$$\leq\mathbf{P}\left\{\int_0^T\left|\int_0^S f_{st}^N W^1(\mathrm{d}s,t)\right|\mathrm{d}t>C\right\}+\mathbf{P}\left\{\int_0^T\left|\int_0^S (f_{st}-f_{st}^N)W^1(\mathrm{d}s,t)\right|\mathrm{d}t>0\right\}$$

$$\leq\frac{T}{C^2}\int_0^T\int_0^S\mathbf{E}(|f_{st}^N|^2)\,\mathrm{d}s\,\mathrm{d}t+\mathbf{P}\left\{\int_0^T\int_0^S f_{st}^2\,\mathrm{d}s\,\mathrm{d}t>N\right\}$$

$$\leq\frac{NT}{C^2}+\mathbf{P}\left\{\int_0^T\int_0^S f_{st}^2\,\mathrm{d}s\,\mathrm{d}t>N\right\},$$

and this proves (2.32).

Stochastic integrals like $\int_0^S (\int_0^T f_{st} W^2(s, \mathrm{d}t))\, \mathrm{d}s$ can be introduced in a similar way. Moreover, if $\{f_{st}, (s,t) \in I\}$ is a measurable, $\mathscr{F}_{st}$-adapted process, such that

$$\mathbf{P}\left\{\int_0^S \int_0^T f_{st}^2 \,\mathrm{d}s\, \mathrm{d}t < \infty\right\} = 1,$$

we can define the stochastic integral

$$\int_0^S \int_0^T f_{st} W^1(\mathrm{d}s, t) W^2(s, \mathrm{d}t). \tag{2.33}$$

In fact, these integrals are first defined in the usual way for processes satisfying $\int_0^S \int_0^T \mathbf{E}(f_{st}^2)\, \mathrm{d}s\, \mathrm{d}t < \infty$, and they can be extended by means of the convergence in probability.

2.5. Proposition. *Let $\{X(z), z \in I\}$ be a diffusion process satisfying Hypotheses* I *to* IV. *Then,*

$$M_z = X_z - \int_{R_z} \left[\frac{\partial a_2}{\partial x}(\sigma, \tau, X_{\sigma\tau}) X(\mathrm{d}\sigma, \tau)\, \mathrm{d}\tau + \frac{\partial a_1}{\partial x}(\sigma, \tau, X_{\sigma\tau}) X(\sigma, \mathrm{d}\tau)\, \mathrm{d}\sigma \right.$$
$$\left. + \left(a - \frac{\partial(a_1 a_2)}{\partial x}\right)(\sigma, \tau, X_{\sigma\tau})\, \mathrm{d}\sigma\, \mathrm{d}\tau\right]$$

is both a one-parameter local $\mathscr{F}_{st}^1$-martingale and a one-parameter local $\mathscr{F}_{st}^2$-martingale, for each fixed t and s, respectively.

Proof. First, note that the above stochastic integrals are defined according to (2.31) and (2.33).

Then, by Ito's formula, we have

$$X_{st} - \int_0^s a_1(\sigma, t, X_{\sigma t})\, \mathrm{d}\sigma =$$
$$= X_{st} - \int_0^s \left(\int_0^t (D_2 a_1)\, \mathrm{d}\tau + \int_0^t \frac{\partial a_1}{\partial x} B_2^{1/2} W^2(\sigma, \mathrm{d}\tau)\right) \mathrm{d}\sigma$$
$$= X_{st} - \int_{R_{st}} \frac{\partial a_1}{\partial x} X(\sigma, \mathrm{d}\tau)\, \mathrm{d}\sigma - \int_{R_{st}} \left(a - a_2 \frac{\partial a_1}{\partial x}\right) \mathrm{d}\sigma\, \mathrm{d}\tau.$$

Therefore,

$$M_{st} = \left(X_{st} - \int_0^s a_1\, \mathrm{d}\sigma\right) - \int_0^t \left(\int_0^s \frac{\partial a_2}{\partial x} B_1^{1/2} W^1(\mathrm{d}\sigma, \tau)\right) \mathrm{d}\tau.$$

For each $t \geq 0$ we know that the first term is a local $\mathscr{F}_{st}^1$-martingale. In order to prove the local $\mathscr{F}_{st}^1$-martingale property of the second term, it suffices to consider the sequence of $\mathscr{F}_{st}^1$-stopping times,

$$T_n = \inf\left\{s : \int_0^t \int_0^s \left(\frac{\partial a_2}{\partial x}\right)^2 B_1(\sigma, \tau, X_{\sigma\tau})\, \mathrm{d}\sigma\, \mathrm{d}\tau \geq n\right\}. \quad \square$$

Our aim is to obtain a local martingale with orthogonal increments associated to X_{st}. To do this we need to introduce other diffusion coefficients. Consider the following additional hypothesis.

Hypothesis V. *There exist continuous functions* $\hat{B}_1(s,t,\tau,x,\xi)$ *and* $\hat{B}_2(s,t,\sigma,x,\eta)$ *defined for* $(s,t)\in I$, $\sigma\in[0,s)$, $\tau\in[0,t)$, x, ξ, η *real numbers, such that*:

(a) B_1 *and* B_2 *are twice continuous differentiable with respect to* x, *and have continuous derivatives with respect to* s, t.

(b) *For any* $\varepsilon>0$,

$$\int_{\substack{|x_2-x|\le\varepsilon\\ |\xi_1-\xi|\le\varepsilon}} (x_2-x)(\xi_1-\xi)P_{(s,\tau)(s+h,t)}(x,\xi,\xi_1,\mathrm{d}x_2)P_1^{\tau}(s,\xi,s+h,\mathrm{d}\xi_1)=$$

$$=\hat{B}_1(s,t,\tau,x,\xi)h+\mathrm{o}(h)$$

and

$$\int_{\substack{|x_1-x|\le\varepsilon\\ |\eta_1-\eta|\le\varepsilon}} (x_1-x)(\eta_1-\eta)P_{(\sigma,t)(s,t+k)}(\eta_1,\eta,x,\mathrm{d}x_1)P_2^{\sigma}(t,\eta,t+k,\mathrm{d}\eta_1)=$$

$$=\hat{B}_2(s,t,\sigma,x,\eta)k+\mathrm{o}(k).$$

(c) $$\lim_{\substack{t\downarrow\tau\\ x\to\xi}}\hat{B}_1(s,t,\tau,x,\xi)=B_1(s,\tau,\xi) \quad \textit{and} \quad \lim_{\substack{s\downarrow\sigma\\ x\to\eta}}\hat{B}_2(s,t,\sigma,x,\eta)=B_2(\sigma,t,\eta).$$

First, we set up some consequences of this hypothesis.

2.6. Proposition. *Let* $\{X(z), z\in I\}$ *be a diffusion process satisfying Hypotheses* I *to* V. *Then*,

$$\mathbf{E}[X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon\,|\,\mathscr{F}_{st}]=$$

$$=\hat{B}_1(s,t,\tau,X_{st},X_{s\tau})\frac{\partial a_2}{\partial x}(s,t,X_{st})hk+\mathrm{o}(hk)$$

$$=(D_2\hat{B}_1)(s,t,\tau,X_{st},X_{s\tau})hk+\mathrm{o}(hk). \tag{2.34}$$

$$\mathbf{E}[X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_k^2(\sigma,t))_\varepsilon\,|\,\mathscr{F}_{st}]=$$

$$=\hat{B}_2(s,t,\sigma,X_{st},X_{\sigma t})\frac{\partial a_1}{\partial x}(s,t,X_{st})hk+\mathrm{o}(hk)$$

$$=(D_1\hat{B}_2)(s,t,\sigma,X_{st},X_{\sigma t})hk+\mathrm{o}(hk). \tag{2.35}$$

(D_1 *and* D_2 *are applied to the variables* (s,t,x).)

$$\mathbf{E}[X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon X(\Delta_k^2(s,t))_\varepsilon\,|\,\mathscr{F}_{st}]=$$

$$=\tfrac{1}{2}\hat{B}_1(s,t,\tau,X_{st},X_{s\tau})\frac{\partial B_2}{\partial x}(s,t,X_{st})hk+\mathrm{o}(hk)$$

$$=B_2(s,t,X_{st})\frac{\partial\hat{B}_1}{\partial x}(s,t,\tau,X_{st},X_{s\tau})hk+\mathrm{o}(hk). \tag{2.36}$$

$$\mathbf{E}[X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_h^1(s,t))_\varepsilon X(\Delta_k^2(\sigma,t))_\varepsilon \,|\, \mathscr{F}_{st}] =$$

$$= \tfrac{1}{2}\hat{B}_2(s,t,\sigma,X_{st},X_{\sigma t})\frac{\partial B_1}{\partial x}(s,t,X_{st})hk + \mathrm{o}(hk)$$

$$= B_1(s,t,X_{st})\frac{\partial \hat{B}_2}{\partial x}(s,t,\sigma,X_{st},X_{\sigma t})hk + \mathrm{o}(hk). \tag{2.37}$$

Proof. Let f_1, f_2 be infinitely differentiable functions with compact support. As in (2.22) the following convergence takes place a.s., and in L^1, due to the dominated convergence theorem,

$$\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[f_2(X)(\Delta_{hk}(s,t))f_1(X)(\Delta_h^1(s,\tau))\,|\,\mathscr{F}_{st}] =$$

$$= \frac{\partial}{\partial x}(D_2 f_2)(s,t,X_{st})f_1'(X_{s\tau})\hat{B}_1(s,t,\tau,X_{st},X_{s\tau}). \tag{2.38}$$

Fix $(s,t)\in I$, $\tau\in[0,t)$ and suppose $X_{st}=x_0$, $X_{s\tau}=\xi_0$. If f_1 and f_2 satisfy $f_2(x)=x$ for $|x-x_0|\le C$ and $f_1(x)=x$ for $|x-\xi_0|\le C$, we obtain

$$\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[f_2(X)(\Delta_{hk}(s,t))f_1(X)(\Delta_h^1(s,\tau))\,|\,\mathscr{F}_{st}] =$$

$$= \hat{B}_1(s,t,\tau,x_0,\xi_0)\frac{\partial a_2}{\partial x}(s,t,x_0).$$

Then, to prove the first equality of (2.34) it suffices to check that

$$\mathbf{E}[(f_2(X)(\Delta_{hk}(s,t))f_1(X)(\Delta_h^1(s,\tau)) - X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon)\,|\,\mathscr{F}_{st}] = \mathrm{o}(hk). \tag{2.39}$$

To prove (2.39) we can just consider the conditional expectation on the set $[A_{hk\varepsilon}\cap\{|X(\Delta_h^1(s,\tau))|\le\varepsilon\}]^c$, and therefore, we have to bound the following two terms:

(i)
$$|\mathbf{E}[(f_2(X)(\Delta_{hk}(s,t))f_1(X)(\Delta_h^1(s,\tau)) - X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon)\times$$

$$\times 1_{\{|X(\Delta_{hk}(s,t))|\le\varepsilon, |X(\Delta_h^1(s,t))|>\varepsilon, |X(\Delta_k^2(s,t))|\le\varepsilon\}}\,|\,\mathscr{F}_{st}]|$$

$$= |\mathbf{E}[[(f_2(X_{s+h,t+k}) - f_2(X_{s+h,t}) - X_{s+h,t+k} + X_{s+h,t})(X_{s+h,\tau} - X_{s\tau})$$

$$\times 1_{\{|X(\Delta_h^1(s,\tau))|\le\varepsilon\}} + (f_2(X_{s+h,t+k}) - f_2(X_{s+h,t}) - X_{s,t+k} + X_{st})$$

$$\times (f_1(X_{s+h,\tau}) - f_1(X_{s\tau}))1_{\{|X(\Delta_h^1(s,\tau))|>\varepsilon\}}]$$

$$\times 1_{\{|X(\Delta_{hk}(s,t))|\le\varepsilon, |X(\Delta_h^1(s,t))|>\varepsilon, |X(\Delta_k^2(s,t))|\le\varepsilon\}}\,|\,\mathscr{F}_{st}]$$

$$\le \text{const.}\; k\,\mathbf{P}\{|X(\Delta_h^1(s,t))|>\varepsilon\,|\,\mathscr{F}_{st}\} = \mathrm{o}(hk).$$

(ii) $$|\mathbf{E}[(f_2(X)(\Delta_{hk}(s,t))f_1(X)(\Delta_h^1(s,\tau))-X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon)\times$$
$$\times 1_{\{|X(\Delta_{hk}(s,t))|\le\varepsilon,|X(\Delta_h^1(s,t))|\le\varepsilon,|X(\Delta_k^2(s,t))|>\varepsilon,|X(\Delta_h^1(s,\tau))|\le\varepsilon\}}\,|\,\mathcal{F}_{st}]|$$
$$=|\mathbf{E}[(f_2(X_{s+h,t+k})-f_2(X_{s,t+k})-X_{s+h,t+k}+X_{s,t+k})(X_{s+h,\tau}-X_{s\tau})$$
$$\times 1_{\{|X(\Delta_{hk}(s,t))|\le\varepsilon,|X(\Delta_h^1(s,t))|\le\varepsilon,|X(\Delta_k^2(s,t))|>\varepsilon,|X(\Delta_h^1(s,\tau))|\le\varepsilon\}}\,|\,\mathcal{F}_{st}]|$$
$$\le \text{const. } h\mathbf{P}\{|X(\Delta_k^2(s,t))|>\varepsilon\,|\,\mathcal{F}_{st}\}+\mathrm{o}(hk)=\mathrm{o}(hk),$$

where the last inequality has been deduced by the same method as in part (ii) of the equation following (2.23).

Finally, the second equality of (2.34) can be proved by considering the following iterated limits in L^1,

$$\lim_{k\downarrow 0}\frac{1}{k}\lim_{h\downarrow 0}\frac{1}{h}\mathbf{E}[X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon\,|\,\mathcal{F}_{st}].$$

The proof of (2.35) follows the same lines.

Now we are going to check (2.36). As above, fix $(s,t)\in I$, $\tau\in[0,t)$ and suppose $X_{st}=x_0$, $X_{s\tau}=\xi_0$. Let f_1, f_2 be infinitely differentiable functions with compact support, such that $f_1(x)=x$ for $|x-\xi_0|\le C$, and $f_2(x)=x^2$ for $|x-x_0|\le C$. Then, from (2.38) we obtain

$$\lim_{h,k\downarrow 0}\frac{1}{hk}\mathbf{E}[f_2(X)(\Delta_{hk}(s,t))f_1(X)(\Delta_h^1(s,\tau))\,|\,\mathcal{F}_{st}]=$$
$$=\left(2a_2+2x\frac{\partial a_2}{\partial x}+\frac{\partial B_2}{\partial x}\right)(s,t,x_0)\hat{B}_1(s,t,\tau,x_0,\xi_0). \tag{2.40}$$

For all $\omega\in A_{hk\varepsilon}\cap\{|X(\Delta_h^1(s,\tau))|\le\varepsilon\}$ such that $X_{st}(\omega)=x_0$ and $X_{s\tau}(\omega)=\xi_0$, and assuming $C\ge 3\varepsilon$, we have (omitting the point (s,t))

$$f_2(X)(\Delta_{hk})f_1(X)(\Delta_h^1(s,\tau))=[X(\Delta_{hk})^2+2X(\Delta_h^1)X(\Delta_k^2)+2X(\Delta_{hk})X(\Delta_h^1)$$
$$+2X(\Delta_{hk})X(\Delta_k^2)+2X_{st}X(\Delta_{hk})]X(\Delta_h^1(s,\tau)). \tag{2.41}$$

The following limit can be checked as before,

$$\mathbf{E}[[f_2(X)(\Delta_{hk})f_1(X)(\Delta_h^1(s,\tau))$$
$$-(X(\Delta_{hk})_\varepsilon^2+2X(\Delta_h^1)_\varepsilon X(\Delta_k^2)_\varepsilon+2X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon$$
$$+2X(\Delta_{hk})_\varepsilon X(\Delta_k^2)_\varepsilon+2X_{st}X(\Delta_{hk})_\varepsilon)X(\Delta_h^1(s,\tau))_\varepsilon]$$
$$\times 1_{A_{hk\varepsilon}^c\cup\{|X(\Delta_h^1(s,\tau))|>\varepsilon\}}\,|\,\mathcal{F}_{st}]$$
$$=\mathrm{o}(hk). \tag{2.42}$$

Furthermore,

$$\mathbf{E}[X(\Delta_{hk})_\varepsilon^2 X(\Delta_h^1(s,\tau))_\varepsilon\,|\,\mathcal{F}_{st}]=\mathrm{o}(hk) \tag{2.43}$$

and

$$\mathbf{E}[X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon \,|\, \mathscr{F}_{st}] = \mathrm{o}(hk). \tag{2.44}$$

The proof of (2.43) is analogous to that of (2.29). To check (2.44), set

$$\begin{aligned}|\mathbf{E}[X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon \,|\, \mathscr{F}_{st}]| &\leqslant \\ \leqslant |\mathbf{E}[X(\Delta_{hk})_\varepsilon \tfrac{1}{2}(X(\Delta_h^1)_\varepsilon - X(\Delta_h^1(s,\tau))_\varepsilon)^2 \,|\, \mathscr{F}_{st}]| \\ + |\mathbf{E}[X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon^2 \,|\, \mathscr{F}_{st}]| + |\mathbf{E}[X(\Delta_{hk})_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon^2 \,|\, \mathscr{F}_{st}]|,\end{aligned}$$

and each term can be bounded as in the proof of (2.30). Finally, from (2.40), (2.41), (2.42), (2.43) and (2.44) we obtain

$$\lim_{h,k\downarrow 0} \frac{1}{hk} \mathbf{E}[X(\Delta_{hk})_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon X(\Delta_k^2)_\varepsilon \,|\, \mathscr{F}_{st}] = \tfrac{1}{2} \frac{\partial B_2}{\partial x}(s,t,X_{st}) \hat{B}_1(s,t,\tau,X_{st},X_{s\tau}).$$

As before, the second equality of (2.36) can be proved by taking the following iterated limits in L^1,

$$\lim_{k\downarrow 0} \frac{1}{k} \lim_{h\downarrow 0} \frac{1}{h} \mathbf{E}[\mathbf{E}[X(\Delta_{hk}(s,t))_\varepsilon X(\Delta_h^1(s,\tau))_\varepsilon \,|\, \mathscr{F}_{s,t+k}] X(\Delta_k^2(s,t))_\varepsilon \,|\, \mathscr{F}_{st}].$$

(2.37) can be proved analogously. □

2.7. Lemma. *Let $f, g : I \times R \to R$ be continuous functions with compact support. Consider the processes*

$$N_{st} = \int_0^s f B_1^{1/2}(\sigma, t, X_{\sigma t}) W^1(\mathrm{d}\sigma, t),$$

$$M_{st} = \int_0^s \int_0^t g B_1^{1/2}(\sigma, \tau, X_{\sigma\tau}) W^1(\mathrm{d}\sigma, \tau) W^2(\sigma, \mathrm{d}\tau),$$

$X(z)$ being a diffusion process satisfying Hypotheses I *to* V. *Then, for each fixed $t \in [0,T]$, those processes are square integrable continuous martingales with respect to the family $\{\mathscr{F}_{st}^1, s \geqslant 0\}$, and if $t_1 < t_2$,*

$$\langle N_{st_1}, M_{st_2} - M_{st_1} \rangle = \int_0^s f(\sigma, t_1, X_{\sigma t_1}) \times \\ \times \left(\int_{t_1}^{t_2} g(\sigma, \tau, X_{\sigma\tau}) \hat{B}_1(\sigma, \tau, t_1, X_{\sigma\tau}, X_{\sigma t_1}) W^2(\sigma, \mathrm{d}\tau) \right) \mathrm{d}\sigma \tag{2.45}$$

and

$$\langle M_{st_1}, M_{st_2} - M_{st_1} \rangle = \int_0^s \left(\int_{t_1}^{t_2} \left(\int_0^{t_1} g(\sigma, \tau', X_{\sigma\tau'}) g(\sigma, \tau, X_{\sigma\tau}) \times \\ \times \hat{B}_1(\sigma, \tau, \tau', X_{\sigma\tau}, X_{\sigma\tau'}) W^2(\sigma, \mathrm{d}\tau') \right) W^2(\sigma, \mathrm{d}\tau) \right) \mathrm{d}\sigma. \tag{2.46}$$

Proof. We are going to demonstrate (2.45). The proof of (2.46) is similar and the details will be omitted.

Consider a decreasing sequence of partitions of $[t_1, t_2]$, whose norm converges to zero: $t_1 = \tau_0^n < \tau_1^n < \cdots < \tau_{k_n}^n = t_2$. Then, if $\{Y_{st}, (s, t) \in I\}$ is a continuous and bounded process, the sums

$$\sum_{i=1}^{k_n} \int_0^s Y_{\sigma\tau_{i-1}}(W^2(\sigma, \tau_i) - W^2(\sigma, \tau_{i-1}))W^1(\mathrm{d}\sigma, \tau_{i-1})$$

converge in L^2 to the stochastic integral $\int_0^s \int_0^t Y_{\sigma\tau} W^1(\mathrm{d}\sigma, \tau) W^2(\sigma, \mathrm{d}\tau)$. In fact, these sums are stochastic integrals of the processes $\sum_{i=1}^{k_n} Y_{s\tau_{i-1}} 1_{(\tau_{i-1}, \tau_i]}(t)$, which converge to Y_{st} in $L^2(I \times \Omega)$, due to the continuity of Y_{st}. Then, we have

$$\begin{aligned}
&\mathbf{E}\Big[(N_{s+h,t_1} - N_{st_1})(M_{s+h,t_2} - M_{s+h,t_1} - M_{st_2} + M_{st_1}) - \int_s^{s+h} f(\sigma, t_1, X_{\sigma t_1}) \times \\
&\quad \times \Big(\int_{t_1}^{t_2} g(\sigma, \tau, X_{\sigma\tau}) \hat{B}_1(\sigma, \tau, t_1, X_{\sigma\tau}, X_{\sigma t_1}) W^2(\sigma, \mathrm{d}\tau)\Big)\, \mathrm{d}\sigma \,|\, \mathscr{F}_{st}\Big] \\
&= \lim_n \sum_{i=1}^{k_n} \mathbf{E}\Big[(N_{s+h,t_1} - N_{st_1}) \int_s^{s+h} g B_1^{1/2}(\sigma, \tau_{i-1}, X_{\sigma\tau_{i-1}}) \\
&\qquad\qquad \times (W^2(\sigma, \tau_i) - W^2(\sigma, \tau_{i-1})) \\
&\qquad\qquad \times W^1(\mathrm{d}\sigma, \tau_{i-1}) \\
&\qquad - \int_s^{s+h} f(\sigma, t_1, X_{\sigma t_1})(g(\sigma, \tau_{i-1}, X_{\sigma\tau_{i-1}}) \\
&\qquad \times \hat{B}_1(\sigma, \tau_{i-1}, t_1, X_{\sigma\tau_{i-1}}, X_{\sigma t_1})(W^2(\sigma, \tau_i) - W^2(\sigma, \tau_{i-1})))\, \mathrm{d}\sigma \,|\, \mathscr{F}_{st}\Big].
\end{aligned}$$

We claim that for any n and $i = 1, \ldots, k_n$ these conditional expectations vanish. Indeed, it suffices to show that, for all $\tau > t_1$, the process

$$\begin{aligned}
&N_{st_1} \int_0^s g B_1^{1/2}(\sigma, \tau, X_{\sigma\tau}) W^1(\mathrm{d}\sigma, \tau) + \\
&\quad - \int_0^s f(\sigma, t_1, X_{\sigma t_1}) g(\sigma, \tau, X_{\sigma\tau}) \hat{B}_1(\sigma, \tau, t_1, X_{\sigma\tau}, X_{\sigma t_1})\, \mathrm{d}\sigma
\end{aligned}$$

is a one-parameter martingale with respect to the family $\{\mathscr{F}_{s\tau}, s \geq 0\}$.

To do this, we will apply Lemma 1.3. Let F and G be continuous functions of $(z, x) \in I \times R$ such that $F'_x = f$ and $G'_x = g$. Then,

$$\mathbf{E}\Big[(N_{s+h,t_1} - N_{st_1}) \int_s^{s+h} g B_1^{1/2}(\sigma, \tau, X_{\sigma\tau}) W^1(\mathrm{d}\sigma, \tau) +$$

$$-\int_s^{s+h} f(\sigma, t_1, X_{\sigma t_1}) g(\sigma, \tau, X_{\sigma\tau}) \hat{B}_1(\sigma, \tau, t_1, X_{\sigma\tau}, X_{\sigma t_1}) \, \mathrm{d}\sigma \,|\, \mathscr{F}_{s\tau}\Big]$$

$$= \mathbf{E}[(F(X_{s+h,t_1}) - F(X_{st_1}))(G(X_{s+h,\tau}) - G(X_{s\tau})) \,|\, \mathscr{F}_{s\tau}]$$

$$-\mathbf{E}\Big[\int_s^{s+h} f(\sigma, t_1, X_{\sigma t_1}) g(\sigma, \tau, X_{\sigma\tau}) \hat{B}_1(\sigma, \tau, t_1, X_{\sigma\tau}, X_{\sigma t_1}) \, \mathrm{d}\sigma \,|\, \mathscr{F}_{s\tau}\Big]$$

$$+ \mathrm{o}(h) = \mathrm{o}(h)$$

where this expression is bounded by a constant times h (due to Hypothesis IV), and the convergence to zero when $h \downarrow 0$ holds in L^1.

So, the lemma is proved. □

2.8. Theorem. *Let $\{X(z), z \in I\}$ be a diffusion process satisfying Hypotheses* I *to* V. *Then, there exists a two-parameter Wiener process W_z (adjoining, if necessary, a new probability space) such that*

$$X_z = \int_{R_z} \Big[a \, \mathrm{d}\sigma \, \mathrm{d}\tau + \frac{\partial a_2}{\partial x} B_1^{1/2} W^1(\mathrm{d}\sigma, \tau) \, \mathrm{d}\tau + \frac{\partial a_1}{\partial x} B_2^{1/2} W^2(\sigma, \mathrm{d}\tau) \, \mathrm{d}\sigma$$

$$+ B_1^{-1/2} B_2^{-1/2} d W^1(\mathrm{d}\sigma, \tau) W^2(\sigma, \mathrm{d}\tau) \Big] + \int_{R_z} B^{1/2} \, \mathrm{d}W_{\sigma\tau}. \tag{2.47}$$

2.9. Remark. Suppose that B_1^{-1} and B_2^{-1} are bounded and continuous functions and set $\beta = B_1^{-1} B_2^{-1} d$, $\alpha_1 = \partial a_2/\partial x - \beta a_2$, $\alpha_2 = \partial a_1/\partial x - \beta a_1$, $\gamma = a - (\partial/\partial x)(a_1 a_2) + \beta a_1 a_2$. Then, if we substitute $B_1^{1/2} W^1(\mathrm{d}\sigma, \tau)$ and $B_2^{1/2} W^2(\sigma, \mathrm{d}\tau)$ from (2.12) and (2.13), respectively, into (2.47), we obtain a stochastic equation for the process X_z with W_z as the driving force, without the need for the families W^1 and W^2 of ordinary Brownian motions. So that (2.47) can be transformed in the following representation of the process X_z,

$$X_{st} = \int_{R_{st}} \alpha_1 X(\mathrm{d}\sigma, \tau) \, \mathrm{d}\tau + \alpha_2 X(\sigma, \mathrm{d}\tau) \, \mathrm{d}\sigma + \beta X(\mathrm{d}\sigma, \tau) X(\sigma, \mathrm{d}\tau) + \gamma \, \mathrm{d}\sigma \, \mathrm{d}\tau$$

$$+ \int_{R_{st}} B^{1/2} \, \mathrm{d}W_{\sigma\tau}.$$

Formally we could say that X_z is a solution of the following stochastic nonlinear partial differential equation,

$$\frac{\partial^2 X}{\partial s \, \partial t} - \alpha_1(s, t, X_{st}) \frac{\partial X}{\partial s} - \alpha_2(s, t, X_{st}) \frac{\partial X}{\partial t} - \beta(s, t, X_{st}) \frac{\partial X}{\partial s} \frac{\partial X}{\partial t} - \gamma(s, t, X_{st}) =$$

$$= B^{1/2}(s, t, X_{st}) \frac{\partial^2 W}{\partial s \, \partial t}.$$

Proof of Theorem 2.8. First, note that, by the Schwarz inequality,

$$
\begin{aligned}
&|\mathbf{E}[X(\Delta_{hk})_\varepsilon X(\Delta_h^1)_\varepsilon X(\Delta_k^2)_\varepsilon \,|\, \mathscr{F}_z]|^2 \leqslant \\
&\quad \leqslant \mathbf{E}[X(\Delta_{hk})_\varepsilon^2 \,|\, \mathscr{F}_z]\mathbf{E}[|X(\Delta_h^1)_\varepsilon X(\Delta_k^2)_\varepsilon|^2 \,|\, \mathscr{F}_z] \\
&\quad = \mathbf{E}[X(\Delta_{hk})_\varepsilon^2 \,|\, \mathscr{F}_z]\mathbf{E}[X(\Delta_h^1)_\varepsilon^2 \,|\, \mathscr{F}_z]\mathbf{E}[X(\Delta_k^2)_\varepsilon^2 \,|\, \mathscr{F}_z].
\end{aligned}
$$

Therefore, we have $d^2 \leqslant BB_1B_2$ and the stochastic integral $\int_{R_z} B_1^{-1/2}B_2^{-1/2}d \times W^1(\mathrm{d}\sigma, \tau)W^2(\sigma, \mathrm{d}\tau)$ is well defined.

Set

$$
\begin{aligned}
Y_z = X_z - \int_{R_z} \Big[& a \,\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{\partial a_2}{\partial x} B_1^{1/2} W^1(\mathrm{d}\sigma, \tau)\,\mathrm{d}\tau + \frac{\partial a_1}{\partial x} B_2^{1/2} W^2(\sigma, \mathrm{d}\tau)\,\mathrm{d}\sigma \\
& + B_1^{-1/2}B_2^{-1/2} d W^1(\mathrm{d}\sigma, \tau)W^2(\sigma, \mathrm{d}\tau)\Big]. \qquad (2.48)
\end{aligned}
$$

We are going to prove that Y_z is a local martingale with orthogonal increments. That means, there exists a sequence Y_z^n of square integrable martingales with orthogonal increments, such that for all $\omega \in \Omega$ (a.s.) there is an $N(\omega)$ with $Y_z^n(\omega) = Y_z(\omega)$ for any $z \in I$, $n \geqslant N(\omega)$.

Fix an integer $n \geqslant 1$, and let f be an infinitely differentiable function satisfying $f(x) = x$ for $|x| \leqslant n$, and $f(x) = 0$ for $|x| \geqslant n+1$. Define

$$
\begin{aligned}
Y_z^n = f(X_z) - \int_{R_z} \Big[& D_1D_2 f \,\mathrm{d}\sigma\,\mathrm{d}\tau + \frac{\partial}{\partial x}(D_2 f) B_1^{1/2} W^1(\mathrm{d}\sigma, \tau)\,\mathrm{d}\tau \\
& + \frac{\partial}{\partial x}(D_1 f) B_2^{1/2} W^2(\sigma, \mathrm{d}\tau)\,\mathrm{d}\sigma + (f'' B_1^{1/2} B_2^{1/2} \\
& + f' B_1^{-1/2}B_2^{-1/2} d) W^1(\mathrm{d}\sigma, \tau)W^2(\sigma, \mathrm{d}\tau)\Big]. \qquad (2.49)
\end{aligned}
$$

From

$$
\begin{aligned}
& f(X_{st}) - \int_0^s D_1 f(\sigma, t, X_{\sigma t})\,\mathrm{d}\sigma = \\
& \quad = f(X_{st}) - \int_0^s \Big(\int_0^t D_2D_1 f \,\mathrm{d}\sigma\,\mathrm{d}\tau + \int_0^t \frac{\partial}{\partial x}(D_1 f) B_2^{1/2} W^2(\sigma, \mathrm{d}\tau) \Big)\,\mathrm{d}\sigma
\end{aligned}
$$

we deduce

$$
\begin{aligned}
Y_{st}^n = & \int_0^s f'(X_{st}) B_1^{1/2}(s, \tau, X_{s\tau}) W^2(s, \mathrm{d}\tau) \\
& - \int_{R_{st}} \Big[\frac{\partial}{\partial x}(D_2 f) B_1^{1/2} W^1(\mathrm{d}\sigma, \tau)\,\mathrm{d}\tau \\
& \qquad + (f'' B_1^{1/2} B_2^{1/2} + f' B_1^{-1/2}B_2^{-1/2} d) W^1(\mathrm{d}\sigma, \tau)W^2(\sigma, \mathrm{d}\tau)\Big].
\end{aligned}
$$

So, for any $t \geqslant 0$, $\{Y^n_{st}, s \in [0, S]\}$ is a square integrable continuous martingale with respect to $\{\mathscr{F}^1_{st}, s \in [0, S]\}$. We want to prove that Y^n_{st} has orthogonal increments, and to do this we will apply Lemma 1.4. Let $z = (s, t) \in I$ and let $K > 0$ be fixed. Using Lemma 2.7 we have

$$\begin{aligned}
A_1 &= \mathbf{E}[f(X)(\Delta^1_h) f(X)(\Delta_{hk}) \,|\, \mathscr{F}^1_{st}] \\
&= h[f'(X_{st}) f'(X_{s,t+k}) \hat{B}_1(s, t+k, t, X_{s,t+k}, X_{st}) \\
&\qquad - f'(X_{st})^2 B_1(s, t, X_{st})] + \mathrm{o}(h),
\end{aligned}$$

$$\begin{aligned}
A_2 &= \mathbf{E}\Big[f(X)(\Delta^1_h) \int_s^{s+h} \int_t^{t+k} \frac{\partial}{\partial x}(D_2 f) B_1^{1/2}(\sigma, \tau, X_{\sigma\tau}) W^1(\mathrm{d}\sigma, \tau)\, \mathrm{d}\tau \,|\, \mathscr{F}^1_{st}\Big] \\
&= h\Big[f'(X_{st}) \int_t^{t+k} \frac{\partial}{\partial x}(D_2 f)(s, \tau, X_{s\tau}) \hat{B}_1(s, \tau, t, X_{s\tau}, X_{st})\, \mathrm{d}\tau\Big] + \mathrm{o}(h),
\end{aligned}$$

$$\begin{aligned}
A_3 &= \mathbf{E}\Big[f(X)(\Delta^1_h) \int_s^{s+h} \int_t^{t+k} \frac{\partial}{\partial x}(f' B_2^{1/2}) B_1^{1/2}(\sigma, \tau, X_{\sigma\tau}) \\
&\qquad \times W^1(\mathrm{d}\sigma, \tau) W^2(\sigma, \mathrm{d}\tau) \,|\, \mathscr{F}^1_{st}\Big] \\
&= h\Big[f'(X_{st}) \int_t^{t+k} \frac{\partial}{\partial x}(f' B_2^{1/2})(s, \tau, X_{s\tau}) \hat{B}_1(s, \tau, t, X_{s\tau}, X_{st}) \\
&\qquad \times W^2(s, \mathrm{d}\tau)\Big] + \mathrm{o}(h),
\end{aligned}$$

$$\begin{aligned}
A_4 &= \mathbf{E}\Big[\Big(\int_s^{s+h} \int_0^t \frac{\partial}{\partial x}(D_2 f) B_1^{1/2}(\sigma, \tau, X_{\sigma\tau}) W^1(\mathrm{d}\sigma, \tau)\, \mathrm{d}\tau\Big) f(X)(\Delta_{hk}) \,|\, \mathscr{F}^1_{st}\Big] \\
&= h\Big[\int_0^t \frac{\partial}{\partial x}(D_2 f)(s, \tau, X_{s\tau})(f'(X_{s,t+k}) \hat{B}_1(s, t+k, \tau, X_{s,t+k}, X_{s\tau}) \\
&\qquad - f'(X_{st}) \hat{B}_1(s, t, \tau, X_{st}, X_{s\tau}))\, \mathrm{d}\tau\Big] + \mathrm{o}(h),
\end{aligned}$$

$$\begin{aligned}
A_5 &= \mathbf{E}\Big[\Big(\int_s^{s+h} \int_0^t \frac{\partial}{\partial x}(D_2 f) B_1^{1/2}(\sigma, \tau, X_{\sigma\tau}) W^1(\mathrm{d}\sigma, \tau)\, \mathrm{d}\tau\Big) \\
&\qquad \times \Big(\int_s^{s+h} \int_t^{t+k} \frac{\partial}{\partial x}(D_2 f) B_1^{1/2}(\sigma, \tau, X_{\sigma\tau}) W^1(\mathrm{d}\sigma, \tau)\, \mathrm{d}\tau\Big) \,|\, \mathscr{F}^1_{st}\Big] \\
&= h\Big[\int_0^t \int_t^{t+k} \frac{\partial}{\partial x}(D_2 f)(s, \tau, X_{s\tau}) \frac{\partial}{\partial x}(D_2 f)(s, \tau', X_{s\tau'}) \\
&\qquad \times \hat{B}_1(s, \tau', \tau, X_{s\tau'}, X_{s\tau})\, \mathrm{d}\tau\, \mathrm{d}\tau'\Big] + \mathrm{o}(h),
\end{aligned}$$

$$A_6=\mathbf{E}\Big[\Big(\int_s^{s+h}\int_0^t \frac{\partial}{\partial x}(D_2 f)B_1^{1/2}(\sigma,\tau,X_{\sigma\tau})W^1(\mathrm{d}\sigma,\tau)\,\mathrm{d}\tau\Big)$$

$$\times\Big(\int_s^{s+h}\int_t^{t+k}\frac{\partial}{\partial x}(f'B_2^{1/2})B_1^{1/2}(\sigma,\tau,X_{\sigma\tau})W^1(\mathrm{d}\sigma,\tau)W^2(\sigma,\mathrm{d}\tau)\Big)\,|\,\mathscr{F}^1_{st}\Big]$$

$$=h\Big[\int_0^t\int_t^{t+k}\frac{\partial}{\partial x}(D_2 f)(s,\tau,X_{s\tau})\frac{\partial}{\partial x}(f'B_2^{1/2})B_1^{1/2}(s,\tau',X_{s\tau'})$$

$$\times\hat{B}_1(s,\tau',\tau,X_{s\tau'},X_{s\tau})\,\mathrm{d}\tau\,W^2(s,\mathrm{d}\tau')\Big]=\mathrm{o}(h),$$

$$A_7=\mathbf{E}\Big[\Big(\int_s^{s+h}\int_0^t\frac{\partial}{\partial x}(f'B_2^{1/2})B_1^{1/2}(\sigma,\tau,X_{\sigma\tau})W^1(\mathrm{d}\sigma,\tau)W^2(\sigma,\mathrm{d}\tau)\Big)$$

$$\times f(X)(\Delta_{hk})\,|\,\mathscr{F}^1_{st}\Big]$$

$$=h\Big[\int_0^t\frac{\partial}{\partial x}(f'B_2^{1/2})(s,\tau,X_{s\tau})(f'(X_{s,t+k})\hat{B}_1(s,t+k,\tau,X_{s,t+k},X_{s\tau})$$

$$-f'(X_{st})\hat{B}_1(s,t,\tau,X_{st},X_{s\tau}))W^2(s,\mathrm{d}\tau)\Big]+\mathrm{o}(h),$$

$$A_8=\mathbf{E}\Big[\Big(\int_s^{s+h}\int_0^t\frac{\partial}{\partial x}(f'B_2^{1/2})(\sigma,\tau,X_{\sigma\tau})W^1(\mathrm{d}\sigma,\tau)W^2(\sigma,\mathrm{d}\tau)\Big)$$

$$\times\Big(\int_s^{s+h}\int_t^{t+k}\frac{\partial}{\partial x}(D_2 f)B_1^{1/2}(\sigma,\tau,X_{\sigma\tau})W^1(\mathrm{d}\sigma,\tau)\,\mathrm{d}\tau\Big)\,|\,\mathscr{F}^1_{st}\Big]$$

$$=h\Big[\int_0^t\int_t^{t+k}\frac{\partial}{\partial x}(f'B_2^{1/2})(s,\tau,X_{s\tau})\frac{\partial}{\partial x}(D_2 f)(s,\tau',X_{s\tau'})$$

$$\times\hat{B}_1(s,\tau',\tau,X_{s\tau'},X_{s\tau})W^2(s,\mathrm{d}\tau)\,\mathrm{d}\tau'\Big]+\mathrm{o}(h),$$

$$A_9=\mathbf{E}\Big[\Big(\int_s^{s+h}\int_0^t\frac{\partial}{\partial x}(f'B_2^{1/2})B_1^{1/2}(\sigma,\tau,X_{\sigma\tau})W^1(\mathrm{d}\sigma,\tau)W^2(\sigma,\mathrm{d}\tau)\Big)$$

$$\times\Big(\int_s^{s+h}\int_t^{t+k}\frac{\partial}{\partial x}(f'B_2^{1/2})B_1^{1/2}(\sigma,\tau,X_{\sigma\tau})W^1(\mathrm{d}\sigma,\tau)W^2(\sigma,\mathrm{d}\tau)\Big)\,|\,\mathscr{F}^1_{st}\Big]$$

$$=h\Big[\int_0^t\int_t^{t+k}\frac{\partial}{\partial x}(f'B_2^{1/2})(s,\tau,X_{s\tau})\frac{\partial}{\partial x}(f'B_2^{1/2})(s,\tau',X_{s\tau'})$$

$$\times\hat{B}_1(s,\tau',\tau,X_{s\tau'},X_{s\tau})W^2(s,\mathrm{d}\tau)W^2(s,\mathrm{d}\tau')\Big]+\mathrm{o}(h).$$

Therefore,

$$\mathbf{E}[Y^n(\Delta_h^1)Y^n(\Delta_{hk})\,|\,\mathscr{F}^1_{st}]=$$

$$=h(A_1-A_2-A_3-A_4+A_5+A_6-A_7+A_8+A_9)+\mathrm{o}(h).$$

Here o(h) means that

$$\sup_{\omega}\sup_{h>0}\frac{1}{h}|\mathbf{E}[Y^n(\Delta_h^1)Y^n(\Delta_{hk})\,|\,\mathscr{F}_{st}^1]| \text{ is bounded by some constant,}$$

and, moreover,

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbf{E}[Y^n(\Delta_h^1)Y^n(\Delta_{hk})\,|\,\mathscr{F}_{st}^1] \text{ exists a.s.}$$

Furthermore, taking into account condition (c) of Hypothesis V, we obtain

$$\begin{aligned}A_1 &= f'(X_{st})[f'(X_{s,t+k})\hat{B}_1(s,t+k,t,X_{s,t+k},X_{st}) - f'(X_{st})B_1(s,t,X_{st})]\\ &= f'(X_{st})\bigg[\int_t^{t+k}\frac{\partial}{\partial x}(f'\hat{B}_1)(s,\tau,t,X_{s\tau},X_{st})B_2^{1/2}(s,\tau,X_{s\tau})W^2(s,\mathrm{d}\tau)\\ &\qquad + \int_t^{t+k} D_2(f'\hat{B}_1)(s,\tau,t,X_{s\tau},X_{st})\,\mathrm{d}\tau\bigg] = A_2 + A_3.\end{aligned}$$

Indeed, from (2.34) to (2.37) we deduce that for all $(s,t)\in I$, $\tau\in[0,t)$ and $x,\xi\in\boldsymbol{R}$, the following equalities hold,

$$\frac{\partial}{\partial x}(f'\hat{B}_1)(s,t,\tau,x,\xi)B_2^{1/2}(s,t,x) = \frac{\partial}{\partial x}(f'B_2^{1/2})(s,t,x)\hat{B}_1(s,t,\tau,x,\xi), \tag{2.50}$$

$$D_2(f'\hat{B}_1)(s,t,\tau,x,\xi) = \frac{\partial}{\partial x}(D_2 f)(s,t,x)\hat{B}_1(s,t,\tau,x,\xi). \tag{2.51}$$

In the same way we obtain $A_4 = A_5 + A_6$ and $A_7 = A_8 + A_9$. So, by Lemma 1.4, Y_{st}^n has orthogonal increments.

Now, we are going to show that the process

$$\xi_{st} = (Y_{st}^n)^2 - \int_{R_{st}} [(f')^2 B + (f'')^2 B_1 B_2(\sigma,\tau,X_{\sigma\tau})]\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

is a martingale. It suffices to verify that for all $(s,t)\in I$, $\{\xi_{st},\mathscr{F}_{st}^1, s\in[0,S]\}$ and $\{\xi_{st},\mathscr{F}_{st}^2, t\in[0,T]\}$ are one-parameter martingales, and here we will use Lemma 1.3. First note that

$$\sup_{h>0}\frac{1}{h}|\mathbf{E}[\xi_{s+h,t}-\xi_{st}\,|\,\mathscr{F}_{st}^1]| \le \sup_{h>0}\frac{1}{h}\mathbf{E}[(Y_{s+h,t}-Y_{st})^2\,|\,\mathscr{F}_{st}] + \text{const.} \le \text{const.},$$

as can be checked by looking over each term in the decomposition (2.49) of Y_{st}^n. Then we compute

$$\begin{aligned}&\lim_{h\downarrow 0}\frac{1}{h}\mathbf{E}[(Y_{s+h,t}^n - Y_{st}^n)^2\,|\,\mathscr{F}_{st}] =\\ &\quad = f'(X_{st})^2 B_1(s,t,X_{st}) + 2\int_0^t\int_0^t\left(\frac{\partial}{\partial x}(D_2 f)(s,\tau,X_{s\tau})\,\mathrm{d}\tau\right.\end{aligned}$$

$$+\frac{\partial}{\partial x}(f'B_2^{1/2})(s,\tau,X_{s\tau})W^2(s,\mathrm{d}\tau)\Big)\Big(\frac{\partial}{\partial x}(D_2f)(s,\tau',X_{s\tau'})\,\mathrm{d}\tau'$$

$$+\frac{\partial}{\partial x}(f'B_2^{1/2})(s,\tau',X_{s\tau'})W^2(s,\mathrm{d}\tau')\Big)1_{\{\tau>\tau'\}}\hat{B}_1(s,\tau,\tau',X_{s\tau'},X_{s\tau})$$

$$-2f'(X_{st})\int_0^t\Big(\frac{\partial}{\partial x}(D_2f)(s,\tau,X_{s\tau})\,\mathrm{d}\tau+\frac{\partial}{\partial x}(f'B_2^{1/2})(s,\tau,X_{s\tau})$$

$$\times W^2(s,\mathrm{d}\tau)\Big)\hat{B}_1(s,t,\tau,X_{st},X_{s\tau}).$$

By Ito's formula we have

$$\begin{aligned}f'(X_{st})^2B_1(s,t,X_{st})&=\int_0^t\frac{\partial}{\partial x}((f')^2B_1)B_2^{1/2}W^2(s,\mathrm{d}\tau)+D_2((f')^2B_1)\,\mathrm{d}\tau\\&=\int_0^t 2f'B_1\frac{\partial}{\partial x}(f'B_2^{1/2})W^2(s,\mathrm{d}\tau)\\&\quad+\int_0^t\Big((f')^2D_2B_1+2f'D_2f'+B_1B_2(f'')^2\\&\qquad+2B_2\frac{\partial B_1}{\partial x}f'f''\Big)\,\mathrm{d}\tau\end{aligned}$$

and, therefore,

$$\begin{aligned}&\lim_{h\downarrow 0}\frac{1}{h}\mathbf{E}[(Y^n_{s+h,t}-Y^n_{st})^2\,|\,\mathscr{F}_{st}]=\\&=\int_0^t\Big((f')^2D_2B_1+2f'D_2f'+B_1B_2(f'')^2\\&\qquad+2B_2\frac{\partial B_1}{\partial x}f'f''-2f'B_1\frac{\partial}{\partial x}(D_2f)\Big)\,\mathrm{d}\tau\\&=\int_0^t((f')^2B+(f'')^2B_1B_2)\,\mathrm{d}\tau.\end{aligned}$$

Thus,

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbf{E}[\xi_{s+h,t}-\xi_{st}\,|\,\mathscr{F}^1_{st}]=0,$$

and Lemma 1.3 implies that $\{\xi_{st},\mathscr{F}^1_{st},s\in[0,S]\}$ is a martingale.

The martingale Y^n_{st} belongs to $\mathscr{M}^2_{cc}$ because this process is bounded in L^4, and

$$\langle Y^n\rangle(s,t)=\int_{R_{st}}\alpha_n(\sigma,\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau\quad\text{where }\alpha_n(s,t)=[(f')^2B+(f'')^2B_1B_2](s,t,X_{st}).$$

By Proposition 1.2 there exists a two-parameter Wiener process W_z^n (modifying, if necessary, the original probability space) such that

$$Y_{st}^n = \int_{R_{st}} \alpha_n(\sigma, \tau)^{1/2} \, \mathrm{d}W_{\sigma\tau}^n.$$

For instance, we can set

$$W_{st}^n = \int_{R_{st}} \alpha_n(\sigma, \tau)^{1/2} \, \mathrm{d}Y_{\sigma\tau}^n + \int_{R_{st}} 1_{\{\alpha_n = 0\}}(\sigma, \tau) \, \mathrm{d}\hat{W}_{\sigma\tau},$$

where $\hat{W}_z$ is a two-parameter Wiener process independent of X_z.

Let us introduce the sets

$$B_n = \left\{\omega : \sup_{z \in I} |X_z(\omega)| \leqslant n\right\} \quad \text{and} \quad A_n = \{\omega : Y_z^n(\omega) = Y_z(\omega) \text{ for all } z \in I\}.$$

The local properties of the stochastic integrals which appear in the definition of Y_z and Y_z^n imply that $\mathbf{P}(B_n - A_n) = 0$. Moreover the sequence B_n increases to Ω. Therefore, $\mathbf{P}(\liminf A_n) = 1$. As a consequence, Y_z is a local martingale with orthogonal increments. Then, if we define $D_{nm} = \{\omega : W_z^n(\omega) = W_z^m(\omega) \text{ for all } z \in I\}$, we obtain $\mathbf{P}(B_n - D_{nm}) = 0$ for all $m \geqslant n$, and, therefore, $\mathbf{P}(\bigcup_n \bigcap_{m \geqslant n} D_{nm}) = 1$. So, $W_z = \lim_n W_z^n$ is a two-parameter Wiener process such that $Y_{st} = \int_{R_{st}} B^{1/2}(\sigma, \tau, X_{\sigma\tau}) \, \mathrm{d}W_{\sigma\tau}$. □

References

[1] R. Cairoli, Une classe de processus de Markov, C.R. Acad. Sc. Paris 273 (1971) A1071–A1074.

[2] R. Cairoli and J.B. Walsh, Stochastic integrals in the plane, Acta Math. 134 (1975) 111–183.

[3] I.I. Gihman and A.V. Skorohod, The Theory of Stochastic Processes III (Springer, Berlin-Heidelberg-New York, 1979).

[4] H. Korezlioglu and P. Lefort, Une propriété Markovienne pour les processus à deux indices, C.R. Acad. Sc. Paris 290 (1980) A555–A558.

[5] H. Korezlioglu and G. Mazziotto, Sur les diffusions multidirectionnelles, C.R. Acad. Sc. Paris 290 (1980) A657–A660.

[6] D. Nualart and M. Sanz, A Markov property for two-parameter Gaussian processes, Stochastica 3 (1) (1979) 1–16.

[7] D. Nualart, Martingales à variation indépendante du chemin, in: H. Korezlioglu, G. Mazziotto and J. Szpirglas, eds., Processus Aléatoires à Deux Indices, Lecture Notes in Math. 863 (Springer, New York, 1981) pp. 128–148.

[8] C. Stricker and M. Yor, Calcul stochastique dépendant d'un paramètre, Z. Wahrsch. Verw. Gebiete 45 (1978) 109–133.

[9] E. Wong and M. Zakai, Martingales and stochastic integrals for processes with a multidimensional parameter, Z. Wahrsch. Verw. Gebiete 29 (1974) 109–122.

[10] M. Zakai, Some classes of two-parameter martingales, Ann. Probab. 9 (1981) 255–265.